COMMISSIE VOOR HYDROLOGISCH ONDERZOEK T.N.O.

COMMITTEE FOR HYDROLOGICAL RESEARCH T.N.O.

VERSLAGEN EN MEDEDELINGEN No. 13

PROCEEDINGS AND INFORMATIONS No. 13

# RECENT TRENDS IN HYDROGRAPH SYNTHESIS

TNO

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**Publication Nr. 13**

**Recent trends in hydrograph synthesis**

## CORRECTIONS

p. 7; 23rd line from top: ". . . . . **review** the recent literature . . . . ."

p. 19; 2nd line from bottom: $f - f_c = aF_p$

p. 39; 2nd line from bottom:

$$= \int_0^t \frac{1}{1} e^{-\tau/k_1} \frac{1}{k_2} e^{-t/k_2} e^{\tau/k_2} d\tau = \frac{1}{k_1 k_2} e^{-t/k_2} \int_0^t e^{\tau \frac{k_1 - k_2}{k_1 k_2}} d\tau =$$

p. 48; 20th line from top: ". . . . . interval **of** N years . . . . ."

p. 50; 19th line from top: ". . . . . rainfall **onto** the line . . . . ."

p. 53; 14th line from top: $q = -kD \frac{\delta y}{\delta x}$

p. 62; Edson, C. G. "**Parameters** for relating . . . . ."

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PROCEEDING OF

TECHNICAL MEETING 21

VERSL. MEDED. COMM. HYDROL. ONDERZ. T.N.O. 13 - THE HAGUE - 1966

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# INTRODUCTION

The papers that are being presented in these proceedings were the subject of lectures delivered at a Technical Meeting of the Committee for Hydrological Research TNO in december 1965. This meeting was organized with a view to confront the relatively young surface water hydrology in the Netherlands with recent developments abroad.

In the greater part of this country people can only live by the grace of a strictly controlled water table and for this reason the study of groundwater flow has a long history in the Netherlands. The discharge of excess water was usually regarded as runoff of ground water through a system of natural channels or ditches. Small wonder that the analytical approach of the rainfall runoff relations is mainly based on our knowledge of groundwater flow and not much attention has been paid to the process of overland flow and the runoff process just below the soil surface. In other parts of the world however the situation is more or less the reverse and the efforts were directed towards the socalled direct runoff whereas the baseflow, which was usually identified with groundwater flow, received hardly any attention.

As was to be expected of two approaches which both have a rational basis, a number of identical trends could be detected in the broad field of their overlap and it was found that one approach was essentially in direct line with the other. It was therefore felt that both approaches could profit by a confrontation and the authors have attempted to provide the links between various methods and to bring forward the fundamental concepts behind them.

Evidently the papers presented here mainly reviews the recent literature but they also include some material that is not generally available to the average worker in this field. It was therefore decided to publish these proceedings in English so that they might also contribute to the discussion of hydrology outside this country.

D. A. KRAIJENHOFF VAN DE LEUR

# I. RAINFALL AND RAINFALL EXCESS

F. E. SCHULZE

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## 1. Introduction

The amount of precipitation falling as rain or snow on a drainage basin may leave this basin in two ways viz. in vaporised or liquid form. That part of the precipitation which leaves the basin in liquid form is called excess rainfall. The runoff process describes how the total rainfall as a function of time ultimately results in the discharge of the amount of excess rainfall as a function of time. In this process two stages can be discerned, firstly the process whereby precipitation results in an amount of excess rainfall and secondly the way in which the excess rainfall ultimately manifests itself in a hydrograph of outflow (the discharge as a function of time). In the first case one is interested in the rainfall excess occurring locally throughout the drainage basin, in the second case in the shape of the hydrograph at the point of outflow. Not always does the total amount of excess rainfall pass the point of outflow of the drainage basin: in some cases part of it may leave the basin as deep seepage.

In the following, attention will be paid only to the rainfall excess/rainfall relation as it occurs at every point of the drainage basin. The way in which the locally formed rainfall excess ultimately results in an outflow hydrograph is not discussed here. It is important to know, however, in what form the excess rainfall is occurring.

To clarify this concept let us consider a small drainage basin where a storm occurs. The following losses may occur:

(a) part of the rainfall remains on leaves and grass blades, and eventually evaporates: these are interception losses,

(b) part of the rainfall evaporates from depressions in the soil surface or infiltrates and is used to neutralize soil moisture deficiency in the upper horizons from where it evaporates or is withdrawn by the plant roots,

(c) part of the rainfall percolates down into the groundwater reservoir raising the level of the water table,

(d) part of the rainfall infiltrates and percolates along horizontal strata to emerge again: this is sub-surface flow or interflow,

(e) part of the rainfall moves overland to the main stream.

As regards runoff, (a) and (b) are total „losses". The remaining portion of the rainfall is the rainfall excess, provided that no capillary rise of water from the groundwater reservoir is considered. So, in case of a deep groundwater table, the water has escaped evapotranspiration when it has reached the groundwater reservoir. This rainfall excess can be further split up into groundwater runoff or base flow (c) and direct runoff caused by (d) and (e), usually considered together. In many cases the main cause of floods is the direct runoff; in these situations the part of the rainfall which percolates into the groundwater reservoir is also considerd as a loss, which is only correct viewed from the angle of direct runoff.

To describe quantitatively the process whereby, starting from a given amount of precipitation a certain amount of excess rainfall is attained, one applies many, often very differing ways of approach. In the following, a survey of these methods illustrated by a number of typical examples is attempted.

The ultimate aim is to reduce a given rainfall histogram to a histogram of excess rainfall. This histogram of excess rainfall serves as a basis for the second part of the runoff process; starting from this histogram one tries to obtain the ultimate outflow hydrograph. This is attempted by developing mathematical models that provide the linkage between measured quantities of precipitation and the resulting quantities of runoff. The latter are equal to the appropriate quantities of excess rainfall transformed but not reduced in the second part of the process. Since as a rule precipitation data are available over much longer periods than runoff observations, it is possible with the aid of a developed mathematical model to construct the rainfall excess histogram over longer periods.

The time interval of the rainfall histogram which has to be considered is often determined by the available precipitation data. In the cases where only daily rainfall data are available it is impossible to get information over periods shorter than 24 hours. In that case one is compelled to consider one or more days as the time interval of the rainfall histogram. The requirements to be made for the choice of a time interval are determined by the characteristics of the process whereby the excess rainfall results in an outflow hydrograph. For practical reasons a minimum time interval can be considered. If for example a time interval is chosen equal to 1/4 to 1/5 of the so called basin lag (time difference between the centre of area of the histogram of excess rainfall and the centre of area of the resulting runoff) then it appears that the time distribution of excess rainfall within this period hardly influences the time distribution of the runoff. So under these circumstances it would be senseless to consider a smaller time interval for the reduction of rainfall to rainfall excess.

As has been stated above, the rainfall excess usually manifests itself as direct runoff or as groundwater runoff. It is characteristic for nearly all kinds of approaches that they confine themselves to either the one or the other form of runoff. The most striking example is the infiltration approach which aims exclusively at determining the direct runoff whereby the rainfall which percolates into the ground water is considered as a loss. For situations where a great part of the rainfall excess leaves the drainage basin as direct runoff this approach is indeed justified since the influence of the groundwater runoff on the shape of the outflow hydrograph in that instance will be insignificant (at least inasmuch as peak discharges are concerned). The base flow then has the character of a correction to the hydrograph or may sometimes be disregarded altogether.

In some cases, however, a deliberate separation is made between base flow and direct runoff i.e. the total rainfall histogram is reduced to a histogram of rainfall that becomes direct runoff and another histogram of rainfall excess that becomes groundwater runoff. In the second phase of the runoff process this separation is taken into account by following both components of the rainfall excess separately on their way to the point of outflow.

A second feature of the models cited is that the second phase of the runoff process (the way in which the rainfall excess is led to discharge) does not influence the first phase (the reduction of rainfall tot rainfall excess). This is why models as developed by CRAWFORD and LINSLEY (1961), HAMON (1963) and MAKKINK and VAN HEEMST (1966) will not be fully discussed.

## 2. THE BOOKKEEPING METHOD OR THRESHOLD CONCEPT

The bookkeeping method is one of the most common methods to establish, starting from a certain amount of rainfall, an amount of rainfall excess. In agriculture, especially in irrigation agriculture, this method has been used for many years (not, as a matter of fact, to establish rainfall excesses, but to calculate the water requirements in cases of supplemental irrigation). Yet this method has been used in hydrology as well, though with less success than in its application with regard to irrigation.

In its most simple form the functioning of the drainage basin with regard to the transformation of rainfall to rainfall excess is represented schematically by one single reservoir with a maximum capacity $S$. The quantity of water present in the reservoir at a certain moment is $S—d$, where $d$ is called the deficit. Now in principle, one starts from the assumption that runoff will occur only if the reservoir is full, in other words when the deficit has been replenished. If the deficit equals $d$, then a precipitation $P$, where $P$ exceeds $d$, will result in an amount of runoff $Q = P — d$. Consequently, the relation $Q$ versus $P$ is for a given value of $d$ a simple straight line parallel to the line $Q = P$ (as in fig. 1).

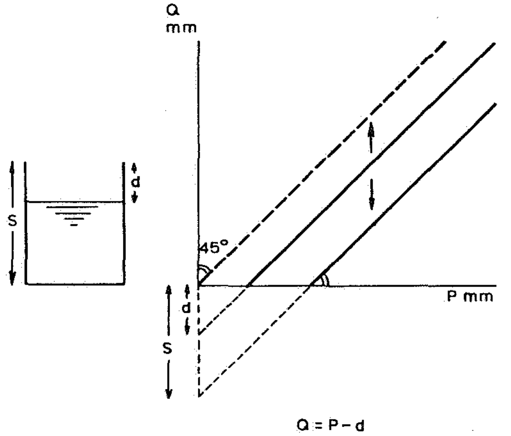


FIG. 1. The bookkeeping method or threshold concept.

Evapotranspiration draws at a known rate from this moisture storage and it is possible now to convert long series of rainfall observations into rainfall excesses if one disposes, for example, of the daily rainfall amounts on the one hand and if on the other a sufficiently exact prediction can be made with regard to the value of the deficit at a given moment. This prediction of the value of the moisture deficit is one of the most characteristic features of the bookkeeping method. Several variations to the above theme have been developed suggesting various methods for the prediction of the value of $d$. Derived purely from irrigation practices, VAN BAVEL (1953) states that the value of $d$ can be calculated by putting actual evapotranspiration equal to potential evapotranspiration until the moment when the storage value $S$ is exhausted. For irrigation purposes this assumption works out satisfactorily in general, probably because in irrigation practice the soil will never dry out to extreme values.

In hydrology, on the contrary, one will usually have to allow for the drying out of the soil which causes the actual evapotranspiration to be lower than the potential evapotranspiration.

For Dutch conditions DE ZEEUW (in print) applies monthly rates of evapotranspiration as determined by ELINK STERK from the water balance of the Haarlemmermeer. However, DE ZEEUW does not start from a certain characteristic value $S$ for the basin, but makes this value depend on the maximum value of the deficit ($d_{max}$) to be calculated from precipitation and evaporation according to ELINK STERK.

Let us say, for example, that the calculated maximum deficit, occurring under Dutch conditions at the beginning of August, is 150 mm, whereas the first increase in runoff is observed mid-October with a calculated deficit of 50 mm. If one does not proceed from the calculated maximum deficit of 150 mm but from a deficit of 150 — 50 = 100 mm at the beginning of August then the moment at which the increase in runoff occurs according to the calculation coincides with the moment at which this happens in reality. By plotting for a series of years the calculated maximum deficit against the value required of coincidence DE ZEEUW obtains the relation:

$$S = a\, d_{max} + b$$

where $a$ and $b$ are constants. So the value of $S$ is not constant but may vary from year to year. The variations in the value of $S$ may be ascribed to the extent to which the real evaporation deviates from the evaporation according to ELINK STERK. As the actual evaporation in a watershed area will generally be smaller than the evaporation according to ELINK STERK, the values of $a$ and $b$ will be such that $S$ is less than $d$.

For calculating the actual evapotranspiration in the Rottegatspolder MAKKINK and VAN HEEMST (1966) have developed a model in which the soil profile is imagined to consist of three zones i.e. the evaporation zone, the transition zone and the groundwater zone. In this rather complicated model the evaporation from the evaporation zone is made proportional to the quantity of water in the evaporation zone which is left for evaporation, or

$$E_a = \frac{S - d}{S} E_p$$

where $E_a$ is the actual and $E_p$ is the potential evaporation and where $S - d$ represents the quantity of water actually present in the evaporation zone and $S$ the maximum possible amount. The model as developed by MAKKINK and VAN HEEMST was basically meant for the calculation of the actual evaporation and not for calculating the rainfall excess. Since the upward flow of moisture from the water table into the root zone is also considered, the model is considerably more complicated than might be presumed from the above. Basically it can however be classified as a bookkeeping method.

KOHLER (1957) has developed a two-level accounting model by allowing for on the one hand a decrease of the actual evapotranspiration if the moisture deficiency in the drainage basin increases and on the other hand the fact that notwithstanding a rather large total moisture deficit the actual evapotranspiration may temporarily have a relatively high value because of moistening of the uppermost layer of the soil profile. Represented diagrammatically this model consists of two reservoirs one placed on top of the other, with maximum capacities of $S_1$ and $S_2$ respectively (fig. 2).

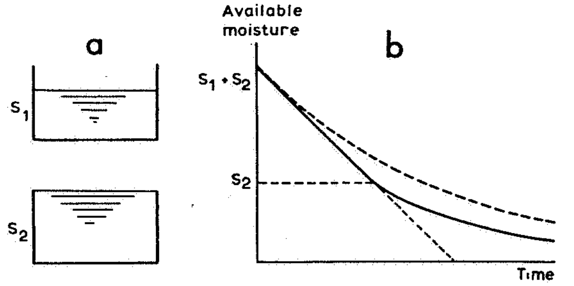


FIG. 2. The two-level accounting model
a: schematically,
b: soil moisture depletion curve.

The upper level of storage represents the upper layer of the soil profile and may be compared with the evaporation zone as visualized by MAKKINK and VAN HEEMST. KOHLER, however, assumes that this reservoir always loses moisture at a rate equal to the rate of potential evapotranspiration. The evapotranspiration from the lower level occurs only when there is no water left in the upper level and it is then proportional to the amount of water available in the second reservoir:

$$E_a = \frac{S_2 - d_2}{S_2} E_p$$

The lower level is replenished only after the moisture deficiency in the upper level has been filled up completely.

VAN SCHILFGAARDE (1965) has used a model in the field of drainage which on the one hand has elements resembling the model developed by MAKKINK and VAN HEEMST (1966) and on the other hand resembles the two-level accounting model of KOHLER (1957). According to WISER and VAN SCHILFGAARDE (1964) the assumption was made that the soil does not hold moisture above a given moisture content, which may be considered to be field capacity, and that the water supplied by precipitation moves further down from a certain level into the soil only if all the soil above this level is at field capacity. If at the outset of a rainfall the moisture content of a soil profile is uniform and below field capacity, it is assumed that after a rainfall the top layer of the soil will be at field capacity, the depth of this layer depending on the amount of water supplied. It was further assumed that the soil i.e. the wettest (top) layer supplies water for evapotranspiration uniformly with depth. When the evapotranspiration taking place from this layer has been such that its moisture content has been reduced to that of the layer below, then the two layers merge into one. Under these assumptions the process of periodic precipitation with daily evapotranspiration is likely to result in the formation of a series of soil layers at moisture contents decreasing with depth. The actual evaporation is made proportional to the moisture content of the upper layer:

$$E_a = \frac{S\ M\ C}{M\ S\ C} E_p$$

where *SMC* is equal to the soil moisture content of the upper layer and *MSC* the maximum moisture storage capacity of the same layer (defined as the difference in moisture content between oven dry soil and field capacity). VAN SCHILFGAARDE (1965) considers for drainage purposes an arbitrary upper limit to be the total soil moisture storage capacity of the whole profile; this is comparable with the value used by WISER (1964) who uses the same approach for watersheds ranging from 460 to 5 sq. miles and considers the value of *S* as a basin parameter. WISER, however, is more interested in total monthly volumes of runoff for the design of farm ponds than in the runoff volumes over shorter periods.

It is clear that the bookkeeping method as described before is strongly related to physical soil properties, particulary to the moisture holding capacity of the soil. If one considers a drainage basin as a whole then there is probably within the basin a great variety of soil conditions. This concept has been represented by KOHLER and RICHARDS (1962) in their method of multi-capacity accounting. Instead of representing the basin as one single reservoir different parts of the area are introduced as separate reservoirs, each reservoir having its own maximum capacity, e.g. 2, 5, 10, 20 inches. The evaporation for each reservoir is put equal to the potential evaporation until the storage is exhausted. For the basin as a whole the rate of soil moisture depletion gradually decreases, as shown in figure 3.

Obviously the bookkeeping method is most suitable in those cases where the natural conditions are in good agreement with the principles of the method i.e. where there is a subsurface removal of the rainfall excess. On the other hand it can be reasoned that if there is a confining horizon in the profile the rainfall excess is being discharged over the land surface after saturation

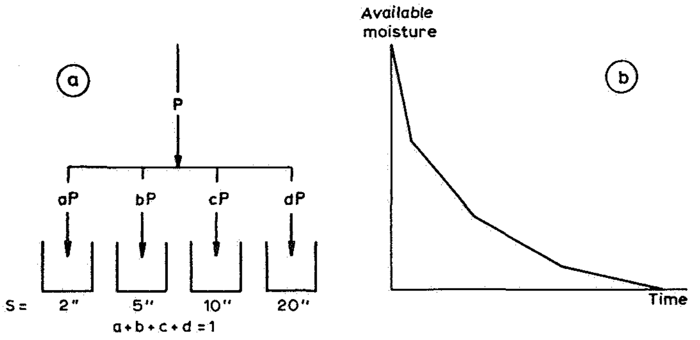


FIG. 3. The multi-capacity accounting model
a: schematically,
b: soil moisture depletion curve.

of the layers above this horizon. In that case the threshold concept could consequently be applied for water flowing over the land surface as well. Applications in this direction in general do not yield satisfactory results since the rainfall intensity per se is not considered. The threshold concept, as a matter of fact, is characterized by the fact that runoff can occur only if the moisture deficiency is replenished completely. The multi-capacity accounting model suggested by KOHLER and RICHARDS already partly eliminates this difficulty. This drawback will be felt especially in situations in which a great part of the rainfall excess is removed over the land surface. For situations in which the total amount of rainfall excess drains through the soil MAKKINK and VAN HEEMST (1966) also take into account discharges that are possible before the moisture deficit in the soil is fully replenished.

The methods mentioned operate on the basis of daily precipitation with the exception of the models developed by VAN BAVEL and by MAKKINK and VAN HEEMST which were not intended for the calculation of rainfall excess as such. For situations in which runoff process takes place mainly through the subsoil this daily interval may be sufficiently small. In places where the runoff occurs as overland flow, however, one should as a rule consider much smaller intervals of time.

Lastly one may wonder in the light of recent investigations in the field of actual evaporation (RIJTEMA, 1965), whether it would be possible to strengthen the physical basis of predicting moisture deficiency.

## 3. GRAPHICAL CORRELATION METHODS

In hydrology frequent use is made of graphical techniques in order to represent the relation between rainfall and rainfall excess. This applies especially in situations where the greater part of the rainfall excess occurs as overland flow, i.e. water which does not penetrate into the soil. Here one is less interested in the physical background of the runoff process but attention is paid almost exclusively to factors that obviously influence the rainfall/rainfall excess relation. Those factors are quantified by means of graphical correlation techniques based on a great number of observations.

One of the most important factors influencing the rainfall/rainfall excess relation is the moisture condition of the drainage basin at the moment the precipitation starts. With the bookkeeping method the influence of this factor was manifested in the value of the moisture deficit $d$. In the graphical method, by contrast, indices are used. The most common index with regard to the moisture condition of the drainage basin is the antecedent precipitation index *API*. The antecedent precipitation index at a certain moment depends exclusively on the precipitation in the preceding period. The *API* may be defined in a great many ways, e.g. as the total precipitation over the preceding 10-day period. In general, however, one would prefer to attribute less weight to $P_{10}$, the rainfall which has, for example fallen 10 days ago, than to $P_1$, the rainfall of the day preceding the day for which the *API* must be determined. Usually this is accomplished by putting $API = P_0 + k\,P_1 + k^2\,P_2 + k^n\,P_n$ $(k < 1)$. This idea is analogous to the assumption that the outflow from a reservoir is proportional to the storage; it corresponds with the bookkeeping method in which the evaporation is proportional to the amount of water left available for evaporation. Arithmetically the *API* defined above has great advantages.

The value of the coefficient $k$ depends on the potential evapotranspiration and consequently on the time of the year. In practice this influence is obviated by including time of year as a variable to be considered. As one is interested in the moisture conditions of the drainage basin it would be equally obvious to consider an „antecedent precipitation minus runoff" index instead of an *API*. Such refinements however are usually not considered as in fact one has to deal with an index and not with a physically measurable factor.

Besides the *API* and time of year one uses in many cases storm duration as a third variable.

Although the graphical method has been frequently applied, recently, for example, by SOCZYNSKA (1963), LUGIER et al. (1963), or BRUNET-MORET (1965) and RODIER and AUVRAY (1965), it is apparent (especially in the U.S.A.) that one tries to develop models with the aid of the available data which preferably have a strong physical base and which can reproduce the observed amounts of rainfall excess as closely as possible.

## 4. Infiltration approaches

Obviously in areas where much overland flow occurs attempts have been made to determine the amount of rainfall excess on the basis of the infiltration capacity of the soil. In these cases the infiltration approach can be applied advantageously, with the reservation of course that the majority of the rainfall excess occurs as overland flow. The principle of the infiltration approach is rather simple, i.e. runoff occurs when the precipitation intensity exceeds the rate of infiltration.

The most straightforward method based on the principle of the infiltration approach is the infiltration index method. As the name implies this method does not so much deal with the infiltration process as such but is a simple index for the infiltration capacity of the basin.

One starts from the assumption that the average infiltration rate in the basin has a constant value during the whole infiltration process. For each storm period one can assume an infiltration index such that the rainfall excess is indeed equal to the measured quantity of runoff. The difficulty of the method lies in the prediction of the value of the infiltration index (also called the basin recharge capacity). If, for instance, the prediction of the infiltration index were made with the aid of the antecedent precipitation index then there is hardly any reason why one should not derive the rainfall excess directly from the antecedent precipitation itself. Apart from this the method has the drawback that it suggests a distribution of the excess rainfall within the storm period which may be incorrect, since the basin recharge capacity may be decreasing in the course of the storm period.

Undoubtedly Horton was the great promotor of the infiltration concept in hydrology. Horton (1939) found from runoff plot experiments that the infiltration rate as a function of time decreases during the infiltration process according to:

$$\frac{df}{dt} = -k\,(f - f_c)$$

where $f$ is the actual infiltration capacity, $f_c$ the minimum infiltration capacity which is reached if the infiltration process continues indefinitely and $k$ a constant. From this the well known Horton-formula may be derived:

$$f = f_c + (f_o - f_c)\,e^{-kt}$$

where $f_0$ is the initial infiltration capacity at the beginning of the infiltration process. The formula assumes an adequate and continuous supply of water to be available for infiltration.

The great advantage of HORTON's method over the infiltration index method will be clear: by expressing the infiltration capacity as a function of time one gains insight into the distribution of the rainfall excess during a storm period, which as has been argued, is particulary important for small drainage basins subjected to rather long periods of rainfall.

Nevertheless the HORTON method presents two difficulties viz. the prediction of the value of $f_o$ at the moment precipitation starts and the phenomenon that there are moments during a storm period when the precipitation intensity is smaller than the infiltration rate. In such a case the infiltration capacity will decrease at a lower rate than is expressed by the HORTON-formula. Probably these two problems dominate the question of whether the use of physically more acceptable infiltration formulas would be preferable to the more or less empirical HORTON-relation, at least for the rainfall runoff process.

To avoid this difficulty arising when one expresses the infiltration capacity as a function of time, HOLTAN (1961) tries to express the infiltration capacity as a function of the remaining volume of potential storage above the confining horizon (this is the volume of pores to be filled before the rate of infiltration becomes constant) and the permeability of the confining horizon. He found the following experimental relationship:

$$f - f_c = a\ F_p^n$$

Here $f$ is the actual infiltration capacity, $f_c$ a constant which depends on the permeability of the confining horizon, comparable with the HORTON-minimum infiltration capacity. The coefficients $a$ and $n$ are constants for a certain soil-vegetation complex, and $F_p$ is the remaining volume of potential storage above the confining horizon (fig. 4). Experimentally HOLTAN found that $n = 1.387$ whereas the value of $a$ varies from case to case.

The great advantage of this approach is of course that one can allow for the fact that during short time intervals in a storm period the precipitation intensity is smaller than the infiltration capacity of the soil. The difficulty of a correct estimation of the value of $F_p$ at the beginning of the storm period remains.

As observed by OVERTON (1964) it is interesting in this respect to note that when $n = 1$, one again arrives at the HORTON-infiltration formula provided that the infiltration process takes place without any interruptions. Under such conditions:

$$f - f_c = a\ F_p$$

$$-\frac{d\ F_p}{dt} = f - f_c \text{ (continuity equation)}$$

from which the HORTON-formula can be derived with $a = k$.

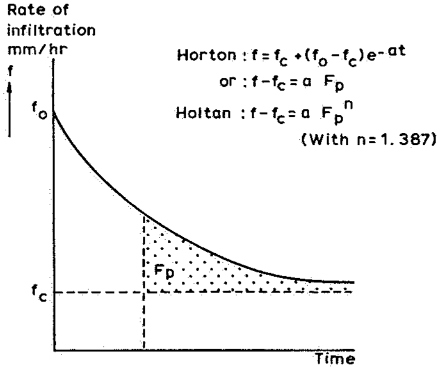


FIG. 4. The infiltration approach.

At the same time it will be seen, starting from the equations:

$$f - f_c = a\,(F_{po} - F_p)^{-1}$$

$$\text{and } f - f_c = \frac{d}{dt}\,(F_{po} - F_p)$$

where $F_{po}$ is the volume of potential storage for $t = o$, one gets the infiltration equation developed by PHILIP (1957) on a physical basis:

$$f = \left(\frac{a}{2}\right)^{\frac{1}{2}} t^{-\frac{1}{2}} + f_c$$

Here the value of $a$ again appears to depend on the initial soil moisture content.

The infiltration approach usually renders good service if very small drainage basins are considered where one is interested in the precipitation deficits during parts of a storm period. The main difficulty of the method, however, remains the prediction of the infiltration capacity at the beginning of a storm. If estimates have to be made for this by correlation analyses, for example with the aid of the antecedent precipitation index, it may be remarked that (especially for the larger basins) the rainfall excess is preferably correlated directly with such an index.

Lastly it may be noted that the rainfall excess as determined by the infiltration approach refers only to overland flow. So in order to obtain the total rainfall excess it may be necessary to add part of the precipitation penetrating into the soil.

## 5. Functional relationships

Starting from the principles of the above-mentioned bookkeeping method or the threshold concept it may be deduced that the relation between precipitation ($P$) and rainfall excess ($Q$) is represented by $Q = P - d$, where $d$ represents the moisture deficit present at a certain moment. Consequently no runoff will occur as long as there exists a moisture deficit. With the data available KOHLER and RICHARDS (1962) find that generally the relation between rainfall and rainfall excess closely approximates the following expression:

$$Q = (P^n + d^n)^{\frac{1}{n}} - d$$

where $d$ is again the moisture deficit at a certain moment and $n$ a coefficient depending on the value of $d$. This relation has the property that as $P$ becomes larger the equation asymptotically approaches the straight line $Q = P - d$. Also the line goes through the origin: $Q = 0$ if $P = 0$ (fig. 5).

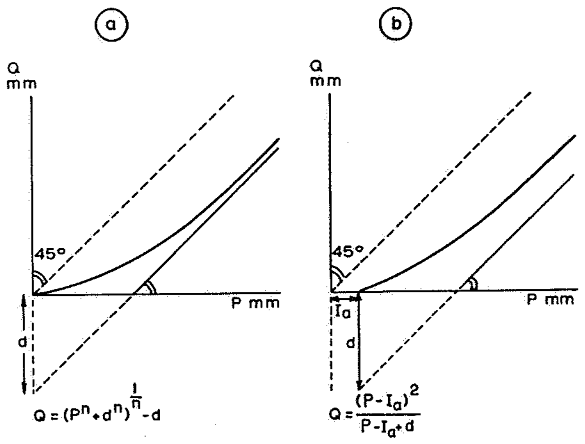


FIG. 5. Runoff equations
a: after KOHLER and RICHARDS (1961),
b: after U.S. Soil Conservation Service (1957).

To achieve a correct estimate of $d$ at the beginning of a precipitation again forms the difficulty of using such a relation. KOHLER and RICHARDS (1962) solved this problem by using the above relation in combination with their multi-capacity accounting model. The principle, of course, is completely the same: instead of calculating the rainfall excess according to $Q = P - d$ (where the threshold concept in practice leads to) one uses the above-mentioned asymptotic relation presuming that it refers to the direct runoff. The value of $d$ at each moment follows from multi-capacity accounting whereas the value of $n$ is calculated with the aid of the following experimental relation:

$n = 2 + 0.5\ d$ where the moisture deficit $d$ should be expressed in inches.

Thus one profits from the advantage of the threshold concept in being able to estimate $d$, yet the disadvantage of this method, being that no runoff can occur as long as there exists a moisture deficit, is avoided.

A much used method developed by the U.S.D.A. Soil Conservation Service (1957) is the so-called curve number method. This method can be classified as a functional relationship as well. The relation between $Q$ and $P$ is determined by starting from a certain initial abstraction ($I_a$), a potential maximum retention ($d$) and the assumption that the ratio of actual retention ($P - I_a - Q$) and potential maximum retention ($d$), equals the ratio of actual runoff ($Q$) and potential maximum runoff ($P - I_a$), or algebraically:

$$\frac{P - I_a - Q}{d} = \frac{Q}{P - I_a}$$

from which follows:

$$Q = \frac{(P - I_a)^2}{(P - I_a) + d}$$

This line asymptotically approaches $Q = P - (I_a + d)$, but does not pass through the origin. The initial abstraction $I_a$ is, on the basis of practical experience, set by the Soil Conservation Service equal to 20% of the maximum retention $d$. The value of $d$ is related to the so-called runoff curve number (from which the name of the method originates) according to the equation:

$$CN = \frac{1000}{10 + d}$$

(for the case when $d$ reaches infinity or $CN = 0$, no runoff occurs at all, while for $d = 0$ or $CN = 100$, the rainfall excess is equal to the precipitation, assuming $I_a = 0.2\ d$).

Again the main difficulty is to make a correct estimate of $CN$ (or of $d$). For practical design purposes the Soil Conservation Service has developed tables for different soil vegetation complexes and antecedent moisture classes. The latter are classed with the aid of an antecedent precipitation index.

Undoubtedly one could mention other examples which can be classified under „functional relationships". The most characteristic features of the methods, however, are demonstrated sufficiently with the above examples in which the endeavour is to avoid the disadvantage of the threshold concept (viz. that runoff can only occur if there is no moisture deficit left in the drainage basin), and to avoid a purely rectilinear relation between $P$ and $Q$ by starting from a rather arbitrary curve asymptotic to a rectilinear relationship.

## 6. Infiltration approach and threshold concept

One of the main difficulties the infiltration approach encounters is how to make a correct estimate of the infiltration rate of the soil at the beginning of a storm period. By relating the infiltration capacity to the remaining volume of potential storage Holtan (1961) could take into account time intervals occurring during the storm period where the precipitation intensity was smaller than the infiltration rate. One difficulty remained however, viz. the estimation of the volume of potential storage at the beginning of the rainfall.

Kohler (1963) has further elaborated the same idea by on the one hand relating the initial infiltration capacity to the existing soil moisture deficit and on the other hand predicting this moisture deficiency with the aid of the multi-capacity accounting technique. Thus this method somewhat resembles the functional relationship methods, but one tries to provide a more physical basis by starting from the infiltration process rather than arbitrarily choosing the form of the runoff equation as is done in the functional relationship methods.

Kohler starts his approach from a slightly modified form of Horton's infiltration equation. This modification is related to the fact that Horton's equation can exclusively be applied for the determination of surface runoff. For determining the total runoff however, one is more interested in the rate at which the soil profile can absorb water (recharge of soil moisture). This capacity rate of absorption must approach zero as precipitation and storm duration increase while the total amount of soil moisture storage is limited. Then in conformity with Horton's approach one gets:

$$f = f_o \, e^{-kt}$$

where as a consequence $f$ is not the infiltration rate but the capacity rate of absorption. From the above formula it may be deduced that at any moment the capacity rate of absorption is proportional to the then existing moisture deficiency $d$:

$$f = kd \text{ or with } t = 0\text{: } f_o = kd_o$$

The equation furthermore shows that the recharge of soil moisture ($r$) equals:

$$r = d\,(1 - e^{-kt}) \text{ or } r = d\,(1 - e^{-\frac{f_o}{d_o} t})$$

Assuming a maximum soil moisture deficiency ($S$) and corresponding maximum rate of soil moisture recharge ($f_k$) from which the value $k = f_k/S$ can be calculated it is possible in principle (for example with the multi-capacity accounting technique) to calculate for every storm period or part thereof the soil moisture recharge and consequently the rainfall excess over that period as

$$Q = P - r$$

where $Q$ is the rainfall excess and $P$ is the precipitation.

A difficulty here is that during the drying out of a saturated soil, for instance, the capacity rate of absorption can reach its maximum value before the soil has reached its maximum soil moisture deficit ($S$). Apart from the total amount of moisture, the capacity rate of absorption also depends on the moisture distribution within the soil profile. In order to take this phenomenon into account, KOHLER makes the initial capacity rate of absorption of an area with a moisture capacity $S$ not only depend on the initial moisture deficiency of this area but also on the moisture deficiency of an area with a moisture capacity of 2 inches. The procedure may be illustrated in figure 6.

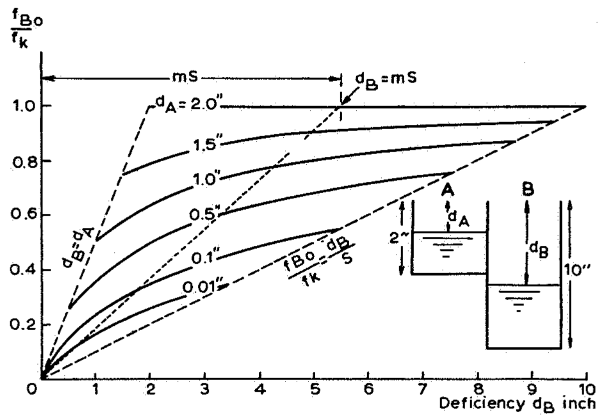


FIG. 6. Threshold concept and infiltration approach.

Take an area A with moisture capacity 2 inches and an area B with moisture capacity $S$. For both areas the maximum capacity rate of absorption is equal to $f_k$. For the area A, the following equation is always applicable:

$$f_A = f_k \frac{d_A}{2}$$

Consequently the above procedure may be applied and the moisture distribution plays no part. The value of $d_A$ can therefore be calculated at any moment.

For the area B, it is assumed that during the wetting phase (starting from $d_B = S$) the following equation is applicable:

$$f_B = f_k \frac{d_B}{S}$$

It is further assumed that for the drying out process in a saturated soil the maximum capacity rate of absorption is reached after a drying out of 2 inches.

Thus the value of $d_B$ as such is inadequate to determine the initial value of $f_B$ ($f_{Bo}$). Therefore the value of $f_{Bo}$ is related to both the values $d_A$ and $d_B$ by the assumption that every combination of $d_A$ and $d_B$ originates from a situation following complete saturation and in which precipitation occurs at a moment where $d_A = 2$ and $d_B = mS$ (where $mS$ is greater than or equal to 2). Now it is possible to demonstrate that starting from such a situation at any moment of the storm period, the following equation applies:

$$\frac{d_A}{2} = \left(\frac{d_B}{mS}\right)^{\frac{mS}{2}}$$

Starting from certain values of $d_A$ and $d_B$ it is possible to reconstruct the hypothetical original situation, i.e. to calculate the value of $mS$. In view of the form of the above equation this should be done graphically. With a deficit of $mS$ greater than or equal to 2 inches and following complete saturation $f_B = f_k$ and as it is generally assumed that $f = kD$, it follows that $f_k = kmS$. During the hypothetical rainfall, as soon as the hypothetical initial deficit $mS$ will have been reduced to $d_B$, the following equation will apply:

$$f_B = kd_B \text{ or } f_B = f_k \frac{d_B}{mS}$$

Thus with the aid of $d_A$ and $d_B$ the calculated value of $f_B$ is assumed to represent the real value of the initial capacity rate of absorption for area B. With the aid of this value the recharge of soil moisture is calculated and as a consequence the quantity of runoff.

KOHLER (1963) applied the above procedure to small intervals during a storm period and determined the values of $d_A$ and $d_B$ for longer periods without rainfall by multi-capacity accounting.

The moisture capacity of 2 inches chosen for area A is arbitrary; under different circumstances another value may be more justified.

The method developed by KOHLER is, despite a number of assumptions, a very interesting one. On the one hand one profits by the advantages offered by the threshold concept with regard to predicting this moisture deficit; on the other hand one attemps to use the advantages of the infiltration approach concerning the predicting of rainfall excesses for parts of a storm period.

## 7. FUNCTIONAL RELATIONSHIPS AND THRESHOLD CONCEPT

We have seen already that KOHLER and RICHARDS (1962) base the calculation of the rainfall excess on a rather arbitrary asymptotic relation between precipitation and rainfall excess and use this relation with their multi-capacity accounting model. It is assumed thereby that the rainfall excess occurs as overland flow. Both DE ZEEUW (in print) and WISER and VAN SCHILFGAARDE (1964) apply functional relationships in order to distinguish between the rainfall excess occurring as overland flow and the rainfall excess occurring as groundwater runoff. As such, this concept differs fundamentally from the model proposed by KOHLER and RICHARDS.

DE ZEEUW starts from the relation:

$$Q_o = m\ (P - I_a)$$

where $Q_0$ is the rainfall excess occurring as overland flow, $P$ the rainfall for the day in question while $m$ and $I_a$ are constants depending on the moisture condition of the basin whereby one distinguishes periods with excess rainfall, periods with an increasing precipitation deficit and periods with a decreasing precipitation deficit.

The value of $Q_0$ so calculated is subtracted from the precipitation $P$. The value $P - Q_0$ is then regarded as input in the water balance model used by DE ZEEUW.

WISER like DE ZEEUW corrects the precipitation for surface runoff before applying the model proposed by him (likewise based on the threshold concept) to the calculation of the rainfall excess. He starts from the earlier mentioned runoff equation of the Soil Conservation Service.

$$Q_o = \frac{(P - 0.2\ d)^2}{P + 0.8\ d}$$

where $Q_0$ represents the amount of surface runoff.

The value of $CN$ which was defined as

$$CN = \frac{1000}{10 + d}$$

is connected by WISER experimentally with the moisture content of the uppermost layer of the soil profile according to the equation

$$CN = 39 + 50 \frac{S\ M\ C}{M\ S\ C}$$

where $S\ M\ C$ represents the soil moisture content of the upper layer of the soil profile and $M\ S\ C$ the maximum storage capacity of the same layer defined as the difference in moisture content between oven dry soil and field capacity.

The value of $S\ M\ C$ results from the accounting model proposed by WISER and VAN SCHILFGAARDE (1964).

Prior to every storm period one knows by application of the water balance method the value of $S\ M\ C$, while $M\ S\ C$ has a certain value.

The remaining quantity of $P - Q_0$ is regarded as the input to the accounting model developed by WISER and VAN SCHILFGAARDE (1964).

## 8. SUMMARY AND CONCLUSION

The foregoing is an attempt to give a review of the various ways of approaching the part of the runoff process which deals with the relation rainfall/rainfall excess. The way in which this rainfall excess occurring throughout the drainage basin ultimately becomes a certain outflow hydrograph is not discussed nor are those models where both phases of the runoff process influence each other. The models referred to above supply the basic data concerning the expected amounts of rainfall excess needed for the practical application of constructions or calculations of hydrographs (such as the method of the unit hydrograph).

An attempt has been made to arrange conveniently the frequently very different approaches. The threshold concept is a theme on which many variations have been worked out. In general the water balance method is particularly appropriate where the rainfall excess runs off via the subsoil. On the other hand the infiltration approach is the obvious method when by far the greater part of the rainfall excess occurs as overland flow. The same applies to graphical correlation techniques and functional relationships, although the application to surface runoff alone is less fundamental in that case. It is characteristic of both methods, however, that they deal with one component of the discharge only, either the surface runoff or the groundwater runoff. Only the infiltration approach and the threshold concept following the method of KOHLER (1963) knowingly take both components together. WISER and VAN SCHILFGAARDE (1964) and DE ZEEUW (in print), on the contrary, make an intended division of the rainfall excess among surface runoff and groundwater runoff.

The introduction of digital computers will no doubt play a dominant rôle in the further development of models concerning the relation rainfall/rainfall excess. On the one hand they create the possibility of shortening the time interval considered; on the other hand it is feasible to compare different models with each other.

The most important development in the use of the digital computer is probably to be found in models developed by CRAWFORD and LINSLEY (1961) in which both phases of the runoff process i.e. the relation rainfall/rainfall excess and rainfall excess-outflow hydrograph are directly connected. This does not mean to say that the models described will consequently lose their significance: the contrary is true. In hydrology the process rainfall excess/outflow hydrograph has always received most attention, especially from the theoretical hydrologists. By relating both components of the runoff process, it may now be expected that the relation rainfall/rainfall excess will ultimately receive also the attention it has been deserving for a long time.

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# II. RUNOFF MODELS WITH LINEAR ELEMENTS

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## 1. Introduction of concepts

The basic assumption underlying the unit hydrograph method is that a drainage basin fed with a unit depth of excess rainfall, uniformly spread both in space and in time, will produce a discharge wave of a certain shape corresponding to the particular duration of excess rainfall that caused the discharge.

The fundamental implications of this assumption have been investigated by DOOGE and NASH. They have provided the theoretical tools for the analysis and evaluation of rainfall/runoff relations developed in the past and have generalized the study of linear runoff systems. In the following presentation a number of basic notions introduced by these hydrologists will be used.

A unit depth of excess rainfall uniformly distributed over a $T$-hour period causes a $T$-hour unit-hydrograph ($TUH$), to be expressed mathematically as $u(T, t)$. The drainage process can be considered as a system in which the input is excess rainfall and the output is the discharge at the basin outlet. It can then be stated that the system transforms a unit block input of duration $T$ into an output to be described as $u(T, t)$.

When the input duration $T$ is reduced the input rate must increase accordingly since the total volume of block input must remain unity. The effect on $u(T, t)$ will be that the peak of this $T$-hour unit hydrograph becomes earlier and higher (fig. 1). It is found however that it gradually merges into its limiting shape, the instantaneous unit hydrograph ($IUH$), indicated as $u(0, t)$. This $IUH$ is the result of an instantaneous input of unit volume into the system.

It may be recalled at this point that in the graphical application of the unit hydrograph method the time distribution of excess rainfall is broken up into unit intervals of steady input rates. These intervals are chosen so short that the corresponding $u(T, t)$ deviates only slightly from the $u(0, t)$ for the basin under consideration. This implies that the deviation from uniformity of the real distribution of input over this „unit storm period" does not affect the shape of the unit hydrograph.

In the theory of linear system analysis (ASELTINE, 1958) the response of a

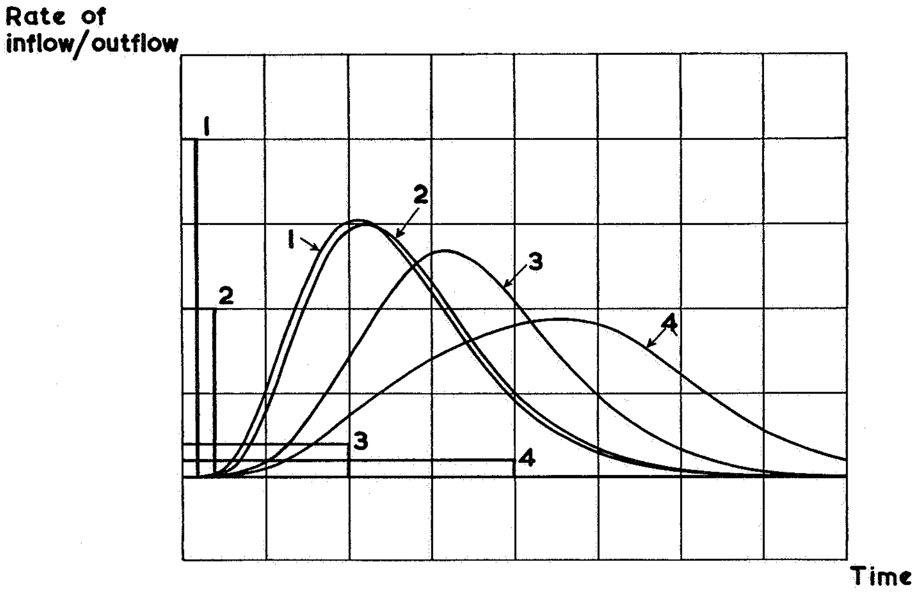


FIG. 1. Shape of the unit hydrograph for various durations of excess rainfall.

system to an infinitesimal input of unit volume or unit impulse ($\delta$-function) is called the indicial response to unit impulse or the impulse response of the system.

The unit hydrograph method presupposes a drainage basin that can be typified by a constant *IUH*, invariant in time ($t$) and independent of preceding events in the runoff process. Consequently any instantaneous input of unit volume will be transformed into the same fundamental „building element" of discharge. In the light of the theory of system analysis such a drainage basin may be considered as a constant coefficient linear system to be described by:

$$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots\ldots A_0 y = x(t) \qquad (1)$$

Such a system indeed has a time-invariant impulse response $h(t)$. This equation also implies that the response to a sum of inputs is identical to the sum of the responses to these inputs applied separately, which essentially is the property of superposition.

Figure 2 shows that a general input can be considered as a succession of infinitesimal instantaneous inputs of volume $x(\tau)d\tau$. Each of these infinitesimal inputs will add its contribution $h\ (t-\tau)x(\tau)d\tau$ to the rate of output $y$ at the time $t$. It follows that the total rate of output equals the sum of all these infinitesimal components:

$$y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau \tag{2}$$

This is the so-called convolution integral which is another characteristic of a linear system, determined by its kernel $h(t-\tau)$.

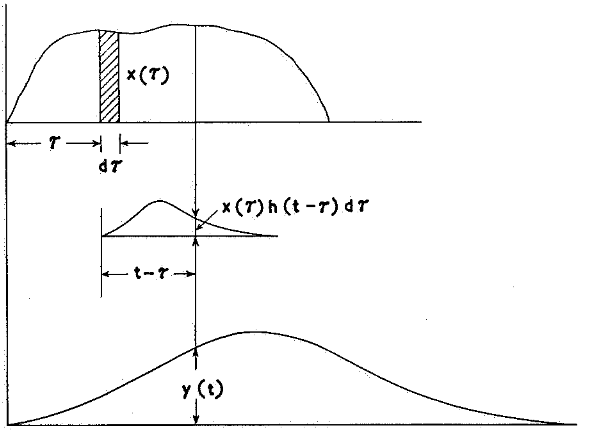


Fig. 2. A linear time-invariant operation.

A reverse reasoning leads to the conclusion that any rainfall/runoff relation that can be described by a constant coefficient linear differential equation satisfies the conditions of the unit hydrograph method. Nash and Dooge have shown that a variety of so called runoff routing procedures can be considered as applications of unit hydrographs of predetermined shapes. In this presentation a similar evaluation procedure will also be applied to a number of rainfall/runoff relations that have been developed in the Netherlands.

## 2. Linear or non-linear

Although the unit hydrograph method has found a world wide application all hydrologists agree that a strictly linear and time-invariant relation between rainfall and runoff cannot exist. The greatest source of non-linearity undoubtedly lies in the procedure of substracting losses and basin recharge from rainfall in order to obtain rainfall excess that will become overland flow and leave the basin as surface runoff at the outlet. According to the usual procedure, base flow, the relatively slow reaction of groundwater runoff, is separated from the outflow hydrograph and the resulting hydrograph of surface runoff is then considered as the result of the operation performed on rainfall excess by the runoff process.

Even if only this transformation into surface runoff is taken into consideration the hydraulic equations which express component flow processes indicate the improbability of strictly meeting the basic requirements of the unit hydrograph method: open channel flow, spatially varied overland flow and many types of storage are non-linear. Another question is, however, to what degree of overall non-linearity may these various non-linear components combine.

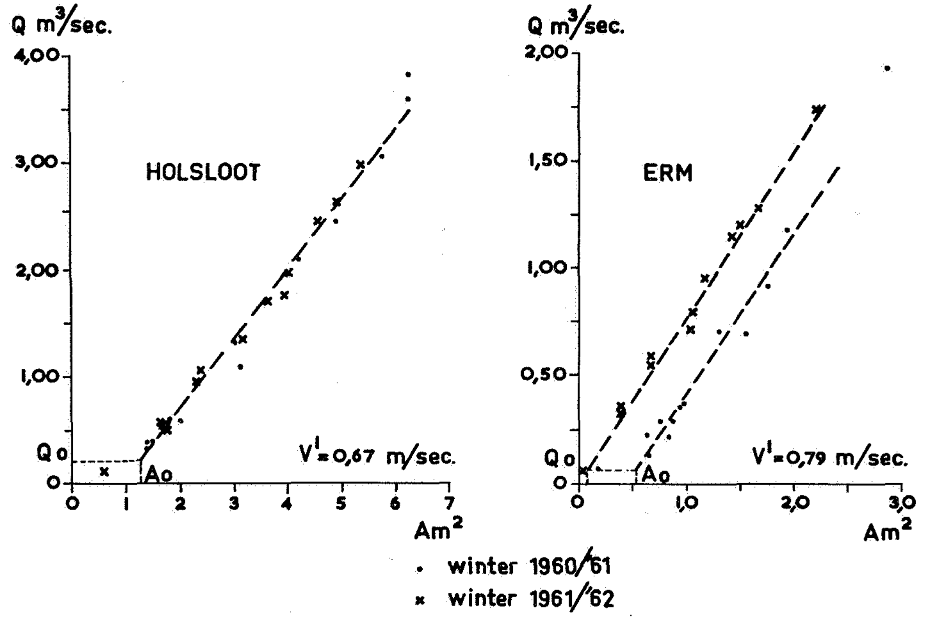


FIG. 3. Linear area-discharge relationships (DE JAGER, 1965).

It should be realized in this connection that channels in nature sometimes show an unexpected linearity as shown in figure 3, taken from DE JAGER's thesis (1965). The reason for this straight line relationship between discharge and wetted cross section could be the increased channel roughness caused by heavy growths of weeds on the channel slopes.

It can also be imagined that overland flow may run through a string of small storages of which some have sub-linear and others supra-linear relations between storage and outflow rate (fig. 4).

Finally it is a well known fact that the occurrence of overbank flow will slow down the propagation of a flood wave whereas its celerity had been increasing with flow during the preceding period of rise. In a channel network overbank flow at different places will occur at different times and here again a source of compensating non-linearities may exist.

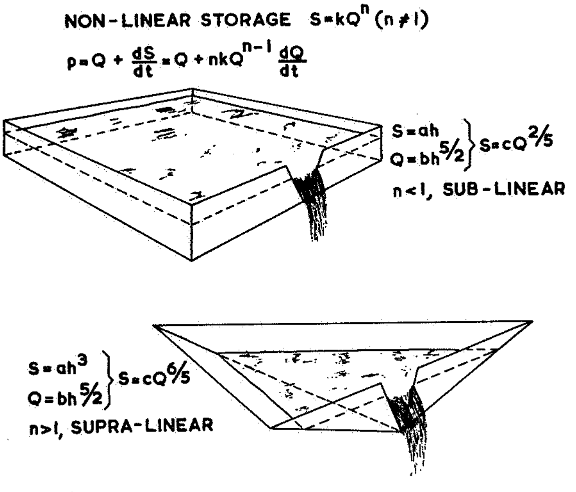


FIG. 4. Different types of non-linear storage.

More research into these components of the runoff process is needed, because the traditional sheet-flow and unsteady flow in prismatic channels can present only a simplified picture of overland flow and flow through the channel system.

Many hydrologists hold the view that the transformation of excess rainfall into surface runoff can in many cases be described by a constant coefficient linear system with a sufficient degree of accuracy.

AMOROCHO and ORLOB (1961), AMOROCHO (1963) and AMOROCHO and HART (1964, 1965) are working along other lines. They state that the assumption of linearity and time invariance must in many cases lead to serious errors and suggest that the principal excuses for maintaining faulty linear procedures are:

(a) Lack of data accurate enough to disprove these methods.

(b) Variable non-uniform distribution of rainfall excess over the drainage basin. Here difficulties will arise for the application of both linear and non-linear systems since in both systems one parameter represents the input and another parameter stands for the output. If no seepage losses occur there is no objection to "lumping" the output. A lumped input however can only represent an input that is uniformly spread over the drainage basin or at least according to a certain fixed pattern. Now it is only too easy to ascribe bad fits of computed and observed hydrographs to the effect of variance in the rainfall distribution and thus pass over another source of errors, the assumption of linearity.

(c) The intricacy of non-linear systems analysis as compared with the relative simplicity of linear systems analysis.

AMOROCHO (1963) distinguishes three types of systems:

1. The linear, time-invariant system where the relation between input $x$ and output $y$ can be expressed as follows:

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau \qquad (3)$$

Here the operator is a fixed function of $t-\tau$, the distance in time between the moment of the infinitesimal input $x(\tau)d\tau$ and the time at which the outflow is considered (fig. 2).

2. The linear, time-variant system, to be expressed by the functional:

$$y(t) = \int_0^t x(\tau)h(\tau,t-\tau)d\tau \qquad (4)$$

This is a more general form of the convolution integral that goes with a variable coefficient linear differential equation:

$$A_n(t)\frac{d^n y}{dt^n} + A_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots\ldots A_0(t)y = x(t) \qquad (5)$$

The operation expressed in the above convolution integral is illustrated in figure 5. It shows the operator as a variable function of $t-\tau$ which function depends on the moment $\tau$ of the infinitesimal input $x(\tau)d\tau$. This means for a runoff system that the *IUH* changes with time; in other words the transformation of a unit of excess rainfall is time-dependent, possibly in the short term (during a storm) or in the long term (with change of season).

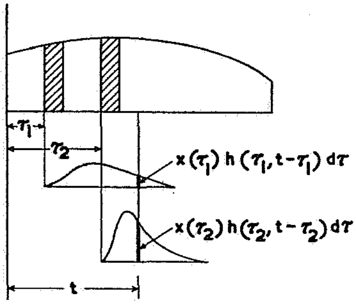


FIG. 5. A linear time-variant operation.

3. The non-linear time-variant system. A rigorous treatment of such systems was not presented, but AMOROCHO suggests that a complex non-linear system which is a polynominal system can be expressed as a series of functionals:

$$y(t) = \int_0^t x(\tau)h(t,\tau)d\tau + \int_0^t \left[ x(\tau_2) \int_0^t x(\tau_1)h(t,\tau_1,\tau_2)d\tau_1 \right] d\tau_2 + \ldots \quad (6)$$

The first term on the right hand side expresses a first approximation by a linear system. The errors caused by the interaction of input elements in the operation are accounted for in the following terms which are convolution integrals of progessively higher dimensions.

A characteristic feature of a non-linear system is that the principle of superposition no longer applies:

$y = x^2$ means $y_1 = x_1{}^2$ and $y_2 = x_2{}^2$ but $y_1 + y_2 \neq (x_1 + x_2)^2$

$y = \sin x$ means $y_1 = \sin x_1$ and $y_2 = \sin x_2$ but $y_1 + y_2 \neq \sin (x_1 + x_2)$

Hydrologic systems are non-linear time-variant: the transformation of rainfall into runoff is dependent both on the season and on the antecedent rainfall. The latter is not only an important factor in the relation between rainfall and excess rainfall but it also determines the initial conditions in non-linear storages and channels when a new wave of runoff is on its way through the drainage basin.

Although it appears that AMOROCHO and his co-workers have made definite advances on their way to a systematic analysis of non-linear systems, they have not yet arrived at a routine procedure for the study of rainfall/runoff relations. Among their achievements to date are impressive contributions to an insight into hydrograph analysis and new alleys of research which they have explored making use of modern methods in the analysis of non-linear systems.

Along with such efforts to develop a fundamental non-linear approach, much work has been done to find new and better linear models to represent the transformation of excess rainfall into discharge. In many cases such linear models have been successful and it seems possible to make use of recently gained insight into sources of non-linearity and introduce local time-variant units and non-linearities into mainly linear models in order to broaden their scope and increase their degree of generality.

A typical feature of such linear models is that they are made up of linear units, either linear storages or linear channels.

## 3. Cascades of linear storages

In the theory of system analysis the symbols $x$ and $y$ usually indicate input and output. In the following, when this theory will be applied to hydrologic systems, however, $p$ will be used for rate of inflow and $q$ will represent the rate of outflow in correspondence with practice in these fields.

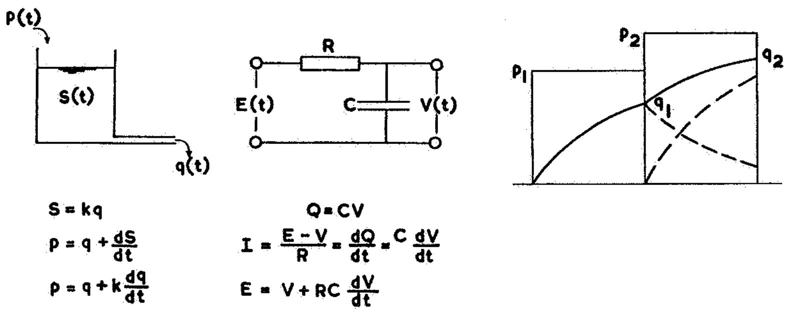


Fig. 6. A hydraulic and an electric analog for a linear storage.

Figure 6 shows a hydraulic linear unit consisting of a cylinder that drains through a capillary and its electrical counterpart, the so-called *RC*-circuit. The differential equation relating output $q(t)$ to an input function $p(t)$ is a constant coefficient linear equation so that this unit will have a constant time-invariant impulse response which can be written as:

$$u(o,t) = \frac{1}{k} \exp(-t/k)$$

In the hydraulic analog in figure 6 the latter equation applies to the depletion of unit storage with $p(t) = o$ for $t > o$.

This simple linear storage model was first introduced by ZOCH (1934). In order to find the "routing equation" for one linear storage the *IUH* can be convoluted with a constant input rate $p_1$:

$$q(t) = p_1 \int_0^t \frac{1}{k} \exp\left(-\frac{t-\tau}{k}\right) d\tau = p_1 e^{-t/k} e^{\tau/k} \Big/_0^t = p_1 \left\{1 - e^{-t/k}\right\} \qquad (7)$$

At the end of the first interval of unit duration:

$$q_1 = p_1 \left(1 - e^{-1/k}\right)$$

According to the superposition principle the outflow rate at the end of the next unit interval of inflow rate $p_2$ is:

$$q_2 = q_1 e^{-1/k} + p_2 \left(1 - e^{-1/k}\right)$$

Thus for a simple linear storage typified by its proportionality factor $k$, the outflow rate at the end of an interval can be derived from the outflow rate at the end of the former interval and the inflow during the considered interval. In general:

$$q_t = q_{t-1} e^{-1/k} + p_t \left(1 - e^{-1/k}\right) \qquad (8)$$

When a linear storage discharges into another linear storage the system may be considered as a cascade of two linear storages with respective proportionality factors $k_1$ and $k_2$. An instantaneous input of unit volume into the first storage causes outflow into the second storage:

$$p_2 = \frac{1}{k_1} \exp\left(-t/k_1\right)$$

The outflow from the system can be found through a convolution of the *IUH* of the second storage with this inflow:

$$u(o,t) = \int_0^t p_2(\tau) \frac{1}{k_2} \exp\left(-\frac{t-\tau}{k_2}\right) d\tau$$

$$= \int_0^t \frac{1}{k_1} e^{-\tau/k_1} \frac{1}{k_2} e^{-t/k_2} e^{\tau/k_2} d\tau = \frac{1}{k_1 k_2} e^{-t/k_2} \int_0^t e^{\tau \frac{k_1-k_2}{k_1 k_2}} d\tau =$$

$$= \frac{1}{k_1-k_2} e^{-t/k_2} \left\{ e^{t \frac{k_1-k_2}{k_1 k_2}} - 1 \right\} = \frac{1}{k_1-k_2} \left\{ e^{-t/k_1} - e^{-t/k_2} \right\} \qquad (9)$$

This expression for the $IUH$ shows that the sequence of the two successive operations does not affect the result: $k_1$ and $k_2$ in equation (9) can be interchanged. This model was published by SUGAWARA and MARUYAMA in 1956.

Following the same procedure expressions for the $IUH$ can now be derived for cascades with increasing numbers of equal storages.

For two equal storages:

$$u(o,t) = \int_0^t \frac{1}{k} e^{-\tau/k} \cdot \frac{1}{k} e^{-t/k} e^{\tau/k} d\tau = \frac{1}{k^2} t e^{-t/k} = \frac{1}{k} \frac{t}{k} e^{-t/k} \qquad (10)$$

For three equal storages:

$$u(o,t) = \int_0^t \frac{1}{k} \frac{\tau}{k} e^{-\tau/k} \frac{1}{k} e^{-t/k} e^{\tau/k} d\tau = \frac{1}{k} \left(\frac{t}{k}\right)^2 \frac{1}{2} e^{-t/k} \qquad (11)$$

For $n$ equal storages:

$$u(o,t) = \frac{1}{k} \left(\frac{t}{k}\right)^{n-1} \frac{1}{(n-1)!} e^{-t/k} \qquad (12)$$

NASH (1957) suggested that such a cascade of $n$ storages is a sufficiently general model of the catchment mechanism and that equation (12) could therefore be taken as the general equation of the $IUH$. To allow non-integral $n$ values NASH substituted the factorial by a Gamma function:

$$u(o,t) = \frac{1}{k} \left(\frac{t}{k}\right)^{n-1} \frac{1}{\Gamma n} e^{-t/k}$$

One may consider the $IUH$ as the frequency distribution of the times of arrival at the outlet of water particles after the instantaneous application of the unit volume of excess rainfall uniformly spread over the drainage basin at zero time. The expectation value $E(\underline{t})$ is the distance in time between the centre of area of the outflow graph and the centre of area of the instantaneous input (which is zero). This mean time of arrival is also called the "lag" of the system.

For the NASH Gamma distribution of (12),

$$E(t) = \int_0^\infty u(o,t) \cdot t \cdot dt = \int_0^\infty t \left(\frac{t}{k}\right)^{n-1} \frac{1}{(n-1)!} e^{-t/k} d\, t/k$$

$$= nk \int_0^\infty \left(\frac{t}{k}\right)^{n} \frac{1}{n!} e^{-t/k} d\, t/k$$

Here the integrand represents another Gamma distribution of order $n + 1$ and its area equals unity. It follows:

$$\text{Lag} = n\,k \tag{13}$$

The spreading of these times of arrival about their mean value $nk$ can be expressed by the second moment of the *IUH* about this centre of area. We thus find the variance of the arrival time:

$$\text{Var}\,(\underline{t}) = E(\underline{t}^2) - \left[E(\underline{t})\right]^2$$

$$E(\underline{t}^2) = \int_o^\infty u(o,t)t^2 dt = \int_o^\infty t^2 \left(\frac{t}{k}\right)^{n-1} \frac{1}{(n-1)!} e^{-t/k} dt/k =$$

$$= k^2 n(n+1) \int_o^\infty \left(\frac{t}{k}\right)^{n+1} \frac{1}{(n+1)!} e^{-t/k} dt/k$$

Here again the integral equals unity because the integrand is a Gamma distribution of the order $n + 2$. It follows that

$$\text{Var}\,(\underline{t}) = k^2n^2 + k^2n - k^2n^2 = k^2n \tag{14}$$

DOOGE (1959) introduced the "linear channel". When a wave passes through a linear channel there is pure translation only and no attenuation occurs. The shape of input and output waves is the same and the lag in a linear channel equals the time of travel of a wave.

DOOGE made the important observation that this linear channel can be considered as a cascade of an infinite number of infinitesimal storages. In this case in the expression lag $= nk$ the number $n$ approaches infinity and $k$ approaches zero. It follows from eq. (14) that the variance approaches zero which means that an instantaneous input of unit volume will cause a similar instantaneous outflow from the system after the expiration of the lag $nk$. If a wave is considered as a succession of instantaneous inflows, it follows that the wave will pass unaltered through the linear channel without change of shape.

NASH (1960) proved that the lag of the *IUH* also represents the distance in time between the centres of area of any inflow graph and the resulting outflow graph (fig. 7). When the inflow graph is broken up in unit intervals of short duration each strip can be substituted by a vertical vector representing its " weight". These strips are subsequently transformed into elements of the outflow wave and each of these elements can also be replaced by a vector representing its "weight".

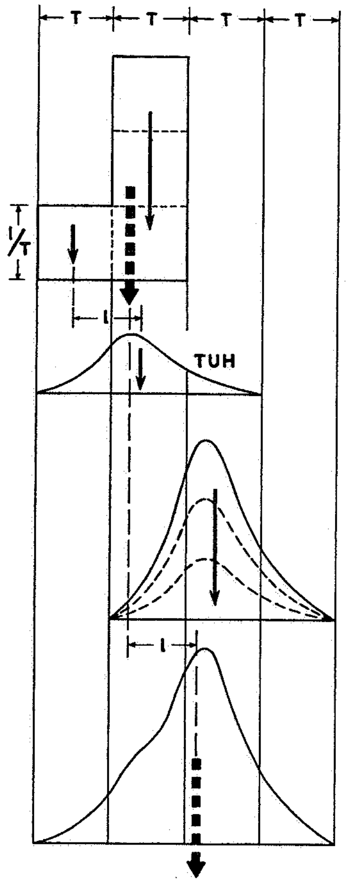


FIG. 7. The lag between inflow and outflow.

It follows that the transformation of inflow into outflow causes the "weight" vector of each element to be shifted in time over the period of lag and therefore the distance between the total "weight" of input and output graphs must also be the lag of the *IUH*.

Another feature of the *IUH* which is frequently used is the time to peak ($t_p$). The *IUH* reaches its maximum when the first derivative equals zero:

$$\frac{d}{dt}\left\{u(\mathrm{o},t)\right\} = 0 = \frac{d}{d(t/k)}\left\{\frac{1}{k}\left(\frac{t}{k}\right)^{n-1}\frac{1}{(n-1)!}e^{-t/k}\right\}$$

$$= \frac{1}{k}\frac{1}{(n-1)!}\left\{(n-1)\left(\frac{t}{k}\right)^{n-2}e^{-t/k} - \left(\frac{t}{k}\right)^{n-1}e^{-t/k}\right\} = 0$$

it follows that $(n-1) - t/k = 0$ and $t_p = (n-1)k$. (15)

A final question is how to derive a $T$-hour unit hydrograph from a known $IUH$. For this purpose NASH (1958) makes use of another indicial response of a system, the so-called $S$-curve, which is the response to a "unit step input".

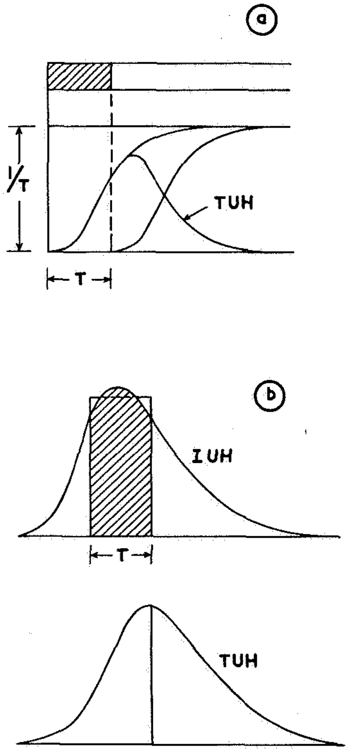


FIG. 8. The graphical construction of a $TUH$
a: from an $S$-curve,
b: from an $IUH$.

This input is a constant inflow of unit intensity starting at zero time. The $S$-curve pictures the growth of the outflow rate to its final unit value (fig. 8). Its mathematical expression follows from the convolution:

$$S_t = \int_0^t u\ (o, t-\tau) d\tau\ .$$

Substituting $t - \tau = \sigma$ and $d\tau = -d\sigma$ (16)

$$S_t = -\int_t^0 u(o,\sigma) d\sigma = \int_0^t u(o,\sigma) d\sigma$$

An $S$-curve starting at the time $T$ can be expressed by

$$S_{t-T} = \int_{o}^{t-T} u(o,\sigma)d\sigma$$

It follows that a block input of duration $T$ and intensity $1/T$ causes the $T$-hour unit hydrograph:

$$u\ (T,t) = \frac{1}{T}\left[\int_{o}^{t} u(o,\sigma)d\sigma - \int_{o}^{t-T} u(o,\sigma)d\sigma\right] = \frac{1}{T}(S_t - S_{t-T}) \qquad (17)$$

$$= \frac{1}{T}\int_{t-T}^{t} u(o,\sigma)d\sigma$$

(valid for $t \geqslant T$. For $t < T$ the lower limit becomes zero).

For a cascade of $n$ equal storages: $u(o,\ t) = \frac{1}{k}\left(\frac{t}{k}\right)^{n-1} \frac{1}{(n-1)!}\ e^{-t/k}$

and the integrals $S_t$ and $S_{t\text{-}T}$ represent incomplete Gamma functions which have been tabulated (Pearson's Tables of Incomplete Gamma Functions).

Figure 8 also shows the graphical interpretation of equation (18): the ordinate $u(T,\ t)$ of the $TUH$ equals the mean ordinate of the $IUH$ over the interval from $t\text{-}T$ to $t$.

After this discussion of the general features of the Gamma distribution which is the mathematical expression of the transformation due to a cascade of linear storages, there follows a review of some linear runoff models.

## 4. Runoff models

In the history of rainfall/runoff models it is possible to discern two developments, one based on the concept of storage between inflow and outflow and the other taking translation as the basic feature of movement to the basin outlet.

### *a. Storage approach*

Lyshede (1955) in his study of Danish watercourses called attention to the various forms of storage through which rainfall excess has to pass on its way to the outlet. He describes the rainfall/runoff relations with a sum of exponential functions which should represent the effect of a cascade of linear storages. Lyshede however adds the observation that any curve can be fairly accurately described by the sum of several exponential functions and therefore the possibilities of a physical interpretation of such models should not be overestimated. In a note he mentions the Gamma distribution as suggested by EDSON (1951) that could well describe the approximate form of unitgraphs.

SATO and MIKKAWA (1956) published a runoff routing method for the transformation of successive hourly rainfall rates into a discharge hydrograph for a small river in Japan. This routing equation is based on the second order Gamma distribution as a fundamental runoff function:

$$u(\mathrm{o},t) = \frac{1}{k}\,\frac{t}{k}\,e^{-t/k}$$

Making use of equation (17) SATO and MIKKAWA's expression for a one hour unit hydrograph can be derived as follows:

$$u(1,t) = \int_{\mathrm{o}}^{t} 1.u(\mathrm{o},\tau)d\tau - \int_{\mathrm{o}}^{t-1} 1.u(\mathrm{o},\tau)d\tau = \int_{t-1}^{t} u(\mathrm{o},\tau)d\tau$$

$$= \int_{t-1}^{t} \frac{1}{k}\,\frac{\tau}{k}\,e^{-\tau/k}\,d\tau = -\int_{t-1}^{t} \frac{\tau}{k}\,d\,e^{-\tau/k}$$

$$= \frac{\tau}{k}e^{-\tau/k}\Big/_{t}^{t-1} - \int_{t-1}^{t} e^{-\tau/k}\,d(-\tau/k)$$

$$= \frac{t-1}{k}e^{-\frac{t-1}{k}} - \frac{t}{k}\,e^{-t/k} - e^{-t/k} + e^{-\frac{t-1}{k}}$$

$$= e^{-\frac{t-1}{k}}\left(\frac{t-1}{k} + 1\right) - e^{-t/k}\left(\frac{t}{k} + 1\right) \quad \text{valid for } t \geqslant 1 \tag{18}$$

For an inflow of $f_o r$ units per hour the flow rate ($t \geqslant 1$) can be written as:

$$q_t = f_o r\left\{e^{-\frac{t-1}{k}}\left(\frac{t-1}{k} + 1\right) - e^{-t/k}\left(\frac{t}{k} + 1\right)\right\} \tag{19}$$

Here $f_o$ is a runoff coefficient.

SATO and MIKKAWA found that a series of terms like (19) could well describe the discharge from the drainage basin caused by a one hour rain of depth $r$:

$$q = F_1(t) + F_2(t) + \dots\dots$$

$$= f_1 r \left\{ e^{-\frac{t-1}{k_1}} \left(\frac{t-1}{k_1} + 1\right) - e^{-t/k_1} \left(\frac{t}{k_1} + 1\right) \right\} +$$

$$f_2 r \left\{ e^{-\frac{t-1}{k_2}} \left(\frac{t-1}{k_2} + 1\right) - e^{-\frac{t}{k_2}} \left(\frac{t}{k_2} + 1\right) \right\} + \dots\dots \quad (20)$$

This runoff system can be simulated by the hydraulic analog in figure 9. The terms of this series of second order Gamma distributions are of decreasing magnitude and SATO and MIKKAWA found that two or three terms gave results of sufficient accuracy.

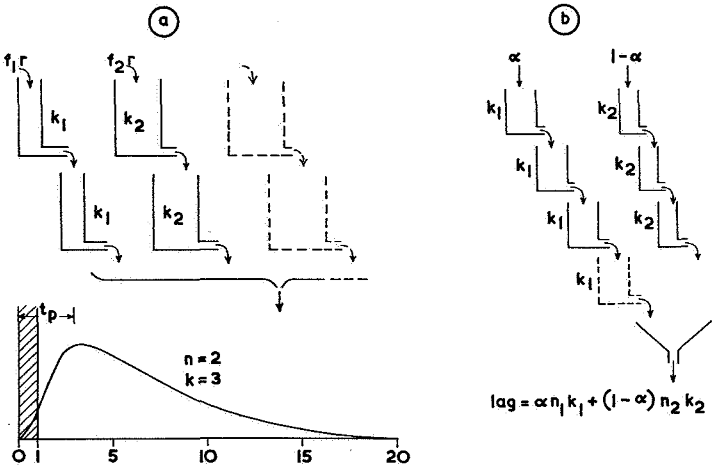


FIG. 9. Parallel cascades of equal storages
a: SATO and MIKKAWA (1956),
b: DISKIN (1964).

In a final note the writers state that the $n$-order Gamma distribution is a suitable element for the characterization of runoff in any river basin.

In their 1956 paper SATO and MIKKAWA indicate that $k_1$ is to be found as the time to peak in the hydrograph. It follows from equation (15) that this is true for the second order Gamma distribution which is the expression for the

*IUH* of each separate branch of two equal storages. The writers apparently assumed that the time of inflow is so short that the hydrograph is practically identical to an instantaneous hydrograph and that the peak caused in the first branch of the model will be dominant. In 1959 TAKENOUCHI describes SATO's method and he states that $k_1$ should be computed from the equation

$$t_p = \frac{e^{1/k_1}}{e^{1/k_1} - 1}$$

which is the time to peak for a one-hour hydrograph of the first branch of the model.

In the example given by SATO and MIKKAWA the time to peak varied between 2 and 5 hours depending on the initial flow rate before the flood wave passed and on the total amount of preceding rainfall. The writers found certain relationships for this dependency so that $k_1$ could be varied in a step by step computation according to these relationships. It should be noted that here the introduction of a non-linear element into a linear system is suggested.

No further details will be given here of the actual curve-fitting by trial and error as presented in the paper; suffice it to draw the following conclusions:

(1) The Gamma distribution was adopted as a basic element of the system response. It is interesting to note that EDSON (1951), NASH (1957) and KALININ and MILYUKOV (1958) arrived at the same conclusion.

(2) As a consequence of (1) the unit hydrograph was represented by a series of Gamma distributions. DOOGE's research has brought him to the same conclusion. In his recently published analysis of linear systems he used Laguerre functions to analyse the input/output response. This method leads to a Gamma distribution series expansion to represent the impulse response of a heavily damped system.

(3) A non-linear feedback procedure of parameters that are determined by the output from the system is indicated. It has already been suggested in the beginning of this paper that this line of enquiry could be pursued further along with AMOROCHO's non-linear approach.

(4) Recently DISKIN (1964) proposed a model which consists of two "NASH-cascades" in parallel which is very similar to SATO and MIKKAWA's model. DISKIN also suggests elements of non-linearity, one of them is that the

lag would vary with the base flow. He does not indicate, however, how his model could account for such non-linearity.

### *b. Translation approach*

According to DOOGE (1959) it was MULVANEY who in 1851 proposed a method that is known as the *rational method.* This method is based on the assumption that the effect of rainfall on the most remote part of the basin takes a certain period, the time of concentration $T_c$, to arrive at the outlet. This time of concentration can either be derived from correlations with basin characteristics or it can be computed from the times of flow in successive "bank-full" reaches of the main channel. It is further assumed that a constant intensity of excess rainfall $Cp$ occurs, uniformly spread over the area A, where $C$ is a runoff coefficient. If this rate of input, a step function, continues until the time of concentration $T_c$ has expired, the excess rainfall that fell on the remotest point of the drainage basin will just begin to cause a reaction at the outflow so that the latter wil have reached its ultimate and maximum rate $Q = Cp\text{A}$.

If it is decided that the design flow rate $Q$ may be exceeded on an average of once in $N$ years, rainfall intensity/duration formulas or graphs are used to find the average rainfall rate $p$ for the period $T_c$ to be exceeded with an average return interval on $N$ years (fig. 10).

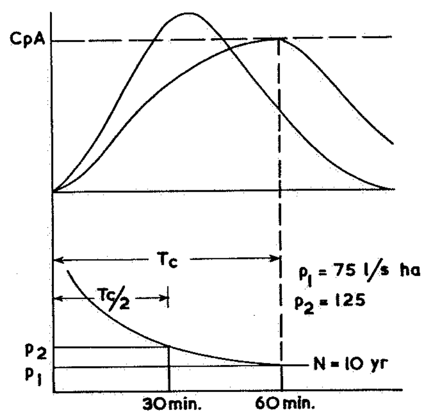


FIG. 10. Rational method.

One fundamental weakness of this method comes out when the growth of $Q$ over the period $T_c$ to its final value $Q = Cp\text{A}$ is considered. This growth can be represented by an $S$-curve the ordinates of which have been multiplied by $CpA$. The shape of this curve is determined by the basin's geometry and topography.

Figure 10 shows the $T_c$ hydrograph and the $\frac{1}{2}T_c$ hydrograph, both caused by rainfall intensities of the same probability $1/N$. Obviously the average rainfall rate $p_2$ with the same recurrence interval of $N$ years but for a period of $\frac{1}{2}T_c$ will result in a higher outflow rate because this rate $p_2$ is considerably higher than the rate $p_1$ for the total time of concentration $T_c$.

The *modified rational method* or *time area method* can be considered as the next step in the translation approach. Using the hydraulic features of the "bank-full" channel system the travel times to the outlet are determined for a number of points in the drainage basin and time contour-lines with equal time intervals are drawn. If it is assumed that an instantaneous excess rain-

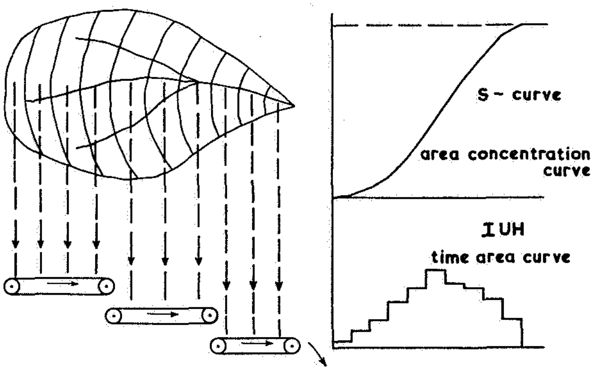


FIG. 11. Modified rational method.

fall of unit depth occurs simultaneously on all points of the basin, the excess rainfall on the elementary area between the time contourlines $t$ and $t + 1$ will arrive at the outlet between $t$ and $t + 1$ and will be represented by the appropriate part of the instantaneous hydrograph situated over this interval. This hydrograph can be called the time area diagram or curve. Dividing all ordinates by the number of surface units A will yield the *IUH* according to the modified rational method.

A number of finite period *TUH*'s are now tried and their ordinates multiplied by the appropriate rates from the rainfall intensity/duration curve in order to find the highest peak flow value (fig. 10). This method certainly shows a marked improvement when compared with the rational method. Of course the method is not restricted to a constant input over the critical period and any design storm can be transformed to an outflow hydrograph.

The topography of the basin may indicate that a certain pattern of areal distribution instead of a uniform rain must be considered as critical. For that case the elementary areas between the time contour lines should be weighted accordingly and this will result in a time area diagram that is adjusted for the variation in rainfall intensity (DOOGE, 1959).

The lag of this linear translation model is the distance in time between the origin and the centre of area of the time area diagram.

Within the scope of this presentation of runoff models with linear elements it is relevant to note that in both the rational method and the modified rational method the translation of excess rainfall is supposed to occur through a system of linear channels. In these channels the travel times are independent of discharge rates.

The channel system can be represented by a system of conveyor belts each moving with its own constant speed independent of the load that is dumped on it. To simplify the picture further the system of conveyors can be replaced by one string of conveyors along the main channel. Each elementary area between two time contour lines dumps its load of excess rainfall into the line of conveyors at the point where it crosses this elementary area. The local translation on the line is slower as the time contour lines are closer together and it follows from continuity reasons that "congestions" of storage will occur at these points. To return to the runoff process this would mean that there is more storage in regions where the velocity of propagation is relatively low. This seems to be natural, but it must be added that the assumption of a constant velocity independent of the discharge rate is not realistic in most cases, since usually the one increases with the other.

NASH (1958) applied the modified rational method to a number of natural drainage basins where actual time distributions of excess rain and outflow rates were available. Comparison of computed and observed hydrographs however, showed a serious overestimation of flood peaks.

### *c. Combined approach*

In a series of papers (1934, 1936, 1937) ZOCH presented a runoff model which consisted of one linear storage that was fed by a rectangular block input of uniform excess rain. He also presented solutions for triangular and elliptic inputs.

These inputs can be considered as the efffect of translation in particular basins (which have the appropriate shape and topography) on an instantaneous excess rainfall. In that case the input diagrams represent the respective time area curves.

Indeed CLARK (1954) used this same idea and presented an *IUH* that was obtained by routing the time area curve through a single linear storage. He first calculated translation times and drew the time contour lines in order to find the time area curve. This curve is usually approximated by a bar diagram (fig. 12) and the successive flow rates of this diagram can be routed

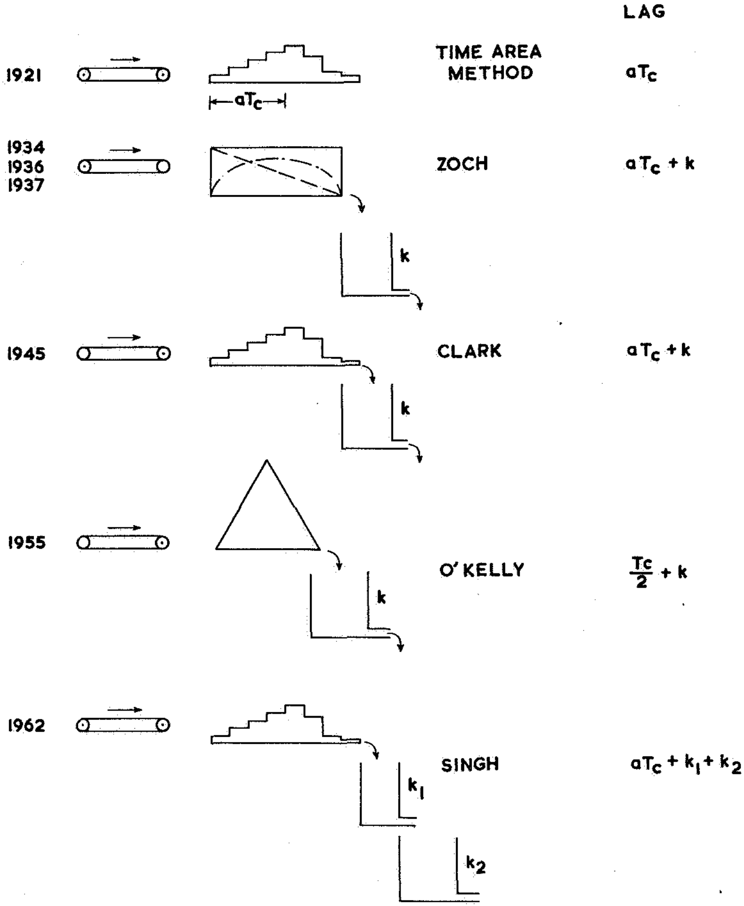


FIG. 12. Combined translation and storage models.

through the linear storage by the use of the routing equation:

$$q_t = q_{t-1}\, e^{-1/k} + p_t\,(1 - e^{-1/k}) \tag{8}$$

O'KELLY (1955) concluded from his study of a number of Irish drainage basins that the smoothing effect of storage on the time area curve was so great that the latter could be replaced by an isosceles triangle without loss of accuracy. The base of this triangle was the time of concentration $T_c$ and its area represented the unit depth of input. O'KELLY routed this input through one linear storage in order to find the *IUH*.

DOOGE (1959) presented a general theory for the linear runoff model. It is based on the assumption that the composite effects of storage and translation in a linear drainage basin can be represented by the transformation performed by a cascade of linear channels connecting equal linear storage elements. The rainfall excess from the elementary areas between successive contour lines is fed into this cascade and subsequently routed through the appropriate length of linear channel and the corresponding number of equal linear storage elements. DOOGE shows that CLARK's and NASH's methods are special cases of his generalized model.

Attention should be drawn to the fact that DOOGE's time area concentration curve represents translation effects which include the delay time due to overbank storage, whereas the classical method of computing travel times to the outlet is based on the assumption of a bankfull channel system.

SINGH (1964) presented a model where the time area curve is routed through two linear storages respectively representing the effects of overland flow and channel flow. Both the second storage parameter $k_2$ and the time concentration $T_c$ vary with the "equivalent instantaneous rainfall excess" which is the ratio of the reconstructed peak discharge and the peak ordinate of the *IUH* used in reconstructing the discharge hydrograph. Since this ratio determines the *IUH* it is a trial and error procedure which introduces a non-linear element into the model. This and a number of other models have been reviewed by VEN TE CHOW (1964).

LAURENSON (1962) discussed a number of runoff models and especially called attention to the fact that separation of translation from attenuation is unreal since any storage produces both. An underlying misconception is to apply the time of travel concept to a "drop of water" whereas the true implication or lag is the time it takes the effect of an element of rainfall excess to reach the outlet. LAURENSON also studied the effect of non-linearities on the relation between rainfall excess and discharge from a drainage basin.

## 5. Parallel developments in the Netherlands

In the Netherlands with its flat topography, its deep soils and long lasting rains of relatively low intensity, surface runoff is not a common phenomenon in natural drainage basins. This was the reason why primary attention was given to the hydrograph of groundwater flow. Little thought was given to unit hydrograph theory since groundwater flow had been explicitly excluded from practical unit hydrograph studies.

In order to arrive at rules that express the relation between rainfall and groundwater runoff efforts were directed towards finding mathematical expressions for the flow system. Considering that the subsoil in this country has been deposited in horizontal layers and the fact that straight parallel drains are frequent, the linearized two dimensional DUPUIT-FORCHHEIMER model was expected to provide a reasonable approximation:

$$\left.\begin{aligned} q &= KD\frac{\delta y}{\delta x} \\ p &= \mu\frac{\delta y}{\delta t} + \frac{\delta q}{\delta x} \end{aligned}\right\} \mu\frac{\delta y}{\delta t} = p + KD\frac{\delta^2 y}{\delta x^2}$$

According to this model, non steady groundwater flow to drains is analogous to one dimensional heat-flow and, following BOUSSINESQ, a number of mathematical techniques that were developed in this field were applied with advantage to the study of groundwater flow.

When applying the classification suggested by AMOROCHO and HART (1964) it might be stated that the study of groundwater runoff should be ranged under physical hydrology since it tries to give a quantitative description of a natural hydrologic system based on the laws of hydrodynamics.

It should be noted that such a model of groundwater flow is very simple as compared with any model that would describe, within a reasonable degree of accuracy, the intricate process of direct runoff. The complete runoff process is a system of interconnected component processes with complicated interactions and it is not yet susceptible to a full quantitative description. Therefore the general type of runoff model as presented in this paper, belongs to the field of system investigations applied to hydrology, what has been called "parametric" hydrology that is aimed solely at finding an input-output relationship that can be used for the reconstruction of past events or the prediction of future events.

Dutch hydrologists have so far been reluctant to leave the safe ground of physical hydrology: they try to stretch their solutions obtained for simplified models to fit hydrologic situations that deviate considerably from their simple basic models. It would seem that in this fitting process a certain amount of subjective judgement is used, based on qualitative and semi-quantitative insight in the role of a number of complicating factors. The main object of hydrologic research in this country has been the improvement of this insight through studies of nature and models.

The unit hydrograph method clearly belongs to the domain of parametric hydrology and moreover it deals exclusively with direct runoff and pays hardly any attention to groundwater flow. For these reasons the theoretical implications of the unit hydrograph method as brought forward by NASH, DOOGE, O'DONNELL and others, at first went by unheeded until it was discovered some years ago that the basic assumptions of linearity and invariance which underlie the unit hydrograph methods are in complete accord with the nature of the simplifying assumptions that have been accepted in order to find analytical solutions for the equations describing the flow of groundwater.

At this moment of discovery it was found that concepts developed in physical groundwater hydrology also played important roles in parametric hydrology. It appeared that these concepts had been developed systematically in parametric hydrology and the results could be used with advantage in the study of groundwater flow from polders and natural drainage basins.

In the following sections a number of Dutch models for rainfall/runoff studies will be discussed and special attention will be given to parallels with parametric hydrology.

EDELMAN (1947) developed equations for the two-dimensional free surface flow of ground water from an infinite stretch of land into a channel where specified level variations and rates of withdrawal occur (fig. 13a). He also noted that the approximating assumption of a constant transmissibility between the free groundwater surface and the impermeable layer, causes the water level variations in the canal to have the same (computed) effect on groundwater flow as appropriate rates of rainfall and evaporation which cause variations of the groundwater level while the water in the channel is kept at the same level.

Although EDELMAN repeatedly uses the superposition principle in his linearized model he derives separate analytical solutions from his equation for the cases of instantaneous and gradual lowering of the water level in the canal. Through the use of the convolution integral the latter solution can be derived simply from the former. This will be shown in the following application of linear model concepts to the flow of ground water to a channel with a fixed level as caused by percolation of rain into the phreatic zone.

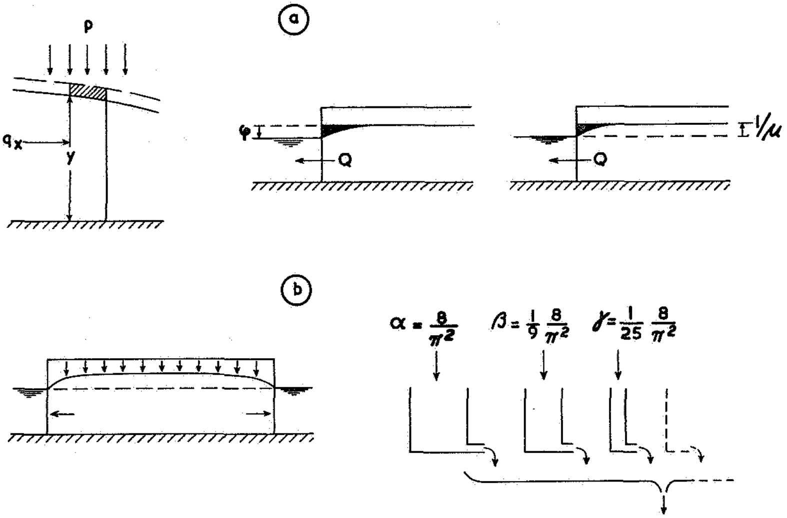


FIG. 13. Models for non-steady groundwater runoff
a: EDELMAN (1947),
b: KRAIJENHOFF (1958).

EDELMAN's equation for one sided flow to a unit length of channel following an instantaneous lowering $\varphi$ of the water level in the channel is:

$$Q(t) = \varphi \frac{1}{\sqrt{\pi}} \sqrt{KD\mu}\; t^{-\frac{1}{2}} \qquad (\varphi << D)$$

here $\mu$ = active porosity and $KD$ = transmissibility.

An instantaneous supply of unit depth of rainfall causes the water table to rise $1/\mu$. The resulting flow to the unit length of channel is

$$u(t) = \frac{1}{\sqrt{\pi}} \sqrt{\frac{KD}{\mu}}\; t^{-\frac{1}{2}}$$

We can now apply the convolution integral in order to find the expression for the increase of groundwater flow as caused by a constant rate $p$ of percolation into the phreatic zone:

$$Q(t) = -\int_{\tau=0}^{\tau=t} \frac{p}{\sqrt{\pi}} \frac{KD}{\mu} (t-\tau)^{-\frac{1}{2}}\, d(t-\tau) = p \frac{2}{\sqrt{\pi}} \sqrt{\frac{KD}{\mu}}\; t^{\frac{1}{2}} \qquad (23)$$

In order to apply this equation to flow from a drainage basin, flow from two sides into a channel must be considered; this means multiplication by a factor 2. Then allowance must be made for the fact that a unit length of channel in a drainage basin only drains a limited stretch of land. The average length of these stretches is the reciprocal of the drainage density $L = A/\Sigma\, l$, where A = basin area and $\Sigma\, l$ = the total length of channels in the basin.

The flow to the channel system expressed as flow per unit area is

$$q_t = \frac{4}{\sqrt{\pi}}\, p \sqrt{\frac{KD}{\mu L^2}\, t} \tag{24}$$

Since the underlying EDELMAN-equation was derived for flow from an infinite stretch of land, this formula is only valid as long as flow to one channel is not being influenced by the presence of the other channels in the system. For a system of equidistant parallel channels this influence can be neglected until a period

$$j = \frac{1}{\pi^2} \frac{\mu L^2}{KD} \tag{25}$$

has expired since the beginning of percolation to a horizontal water table (fig. 14). All factors which determine the nature of the soil and the nature

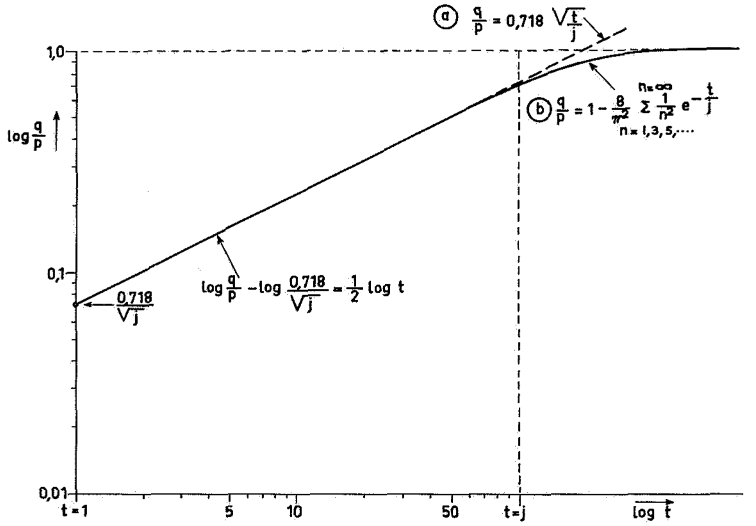


FIG. 14. Growth of outflow rates caused by a step function of inflow according to (a) EDELMAN and (b) KRAIJENHOFF.

and density of the drainage network are incorporated in this "reservoir coefficient" which typifies the drainage situation (KRAIJENHOFF, 1958). In fig. 14 equation (a) is identical with equation (24) and (b) represents the outflow from a stretch of land which has a limited width between two parallel channels (to be discussed in the next section).

Introduction of the reservoir coefficient into eq. (24) indeed yields

$$q_t = \frac{4}{\pi\sqrt{\pi}}\, p \sqrt{t/j} = 0{,}718\ p \sqrt{t/j} \tag{26}$$

If $j$ is expressed in unit intervals the rate of outflow at the end of the third interval for example must be:

$$q_3 = \frac{0{,}718}{\sqrt{j}} \{ p_1 \sqrt{3} + (p_2 - p_1)\sqrt{2} + (p_3 - p_2)\sqrt{1} \}$$

$$= \frac{0{,}718}{\sqrt{j}} \{ p_1 (\sqrt{3} - \sqrt{2}) + p_2(\sqrt{2} - \sqrt{1}) + p_3 \sqrt{1} \}$$

Because of its restricted applicability this simple formula can be used only to calculate groundwater flow caused by intensive short duration inputs of effective rainfall.

GLOVER (1954) studied the falling groundwater table between equidistant parallel ditches or drains following an instantaneous application of a depth $S$ of irrigation water:

$$y(x,t) = \frac{S}{\mu}\frac{4}{\pi} \sum_{n=1,3,5..}^{n=\infty} \frac{1}{n} e^{-n^2 t/j} \sin\frac{n\pi x}{L} \quad \text{with } j = \frac{1}{\pi^2}\frac{\mu L^2}{KD}$$

KRAIJENHOFF (1958) derived from this equation the instantaneous hydrograph of flow to the drainage channels. It can be expressed by:

$$u(o,t) = \frac{8}{\pi^2}\frac{1}{j} \sum_{n=1,3,5..}^{n=\infty} e^{-n^2 t/j} \tag{27}$$

In analogy with the technique of influence lines this "influence function" was integrated in order to find the expression for flow caused by a continuous rate of steady percolation. It is apparent that here the concepts of the *IUH* and the convolution integral were used.

To pursue this parallel, (27) can be written as follows:

$$u(o,t) = \frac{8}{\pi^2}\frac{1}{j}e^{-t/j} + \frac{1}{9}\frac{8}{\pi^2}\frac{9}{j}e^{-9t/j} + \frac{1}{25}\frac{8}{\pi^2}\frac{25}{j}e^{-25t/j} + \ldots.$$

Substituting $k_1 = j$, $k_2 = j/9$ and $k_3 = j/25$ etc.

$$u(o,t) = \frac{8}{\pi^2}\frac{1}{k_1}e^{-t/k_1} + \frac{1}{9}\frac{8}{\pi^2}\frac{1}{k_2}e^{-t/k_2} + \frac{1}{25}\frac{8}{\pi^2}\frac{1}{k_3}e^{-t/k_3} + \ldots \quad (28)$$

It can be shown that eq. (28) expresses the impulse response of a model that consists of parallel linear storages of decreasing magnitude, the respective storages being fed with decreasing parts of the input (fig. 13b). It should be noted that $8/\pi^2\,(1 + \frac{1}{9} + \frac{1}{25} + \ldots) = 1$. A certain similarity with figure 9 is apparant.

In order to find the lag of this model it should be realized that the various parts of input passing through the respective linear storages each undergo their appropriate lag. Since the total input volume is unity, it follows from the first moment about the origin:

$$1.\text{lag} = \frac{8}{\pi^2}\,.\,k_1 + \frac{1}{9}\frac{8}{\pi^2}\,.\,k_2 + \frac{1}{25}\frac{8}{\pi^2}k_3 + \ldots = \frac{8}{\pi^2}j\left\{1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right\} =$$

$$= \frac{8}{\pi^2}\frac{\pi^4}{96}j = \frac{\pi^2}{12}j \quad (29)$$

De Jager (1965) used this model for the synthesis of flood hydrographs of basins in alluvial soils. In cases of flat areas which were well drained by a system of parallel drains he obtained excellent fits with observed hydrographs. Here the drainage situation corresponded closely with the physical basis of the model. With a number of natural basins the agreement proved to be good. In some cases two parallel models were used, one with a relatively small and the other with a relatively large reservoir coefficient.

In his search for a hydrological characteristic for a polder area, Hellinga (1952) found an approximately constant ratio of the daily quantities of water pumped out of the polders and the amounts of rainfall excess that still remained to be pumped out. In other words this is an approximate proportionality of outflow rate and storage (fig. 15).

De Zeeuw and Hellinga observed that storage in a polder area is mainly

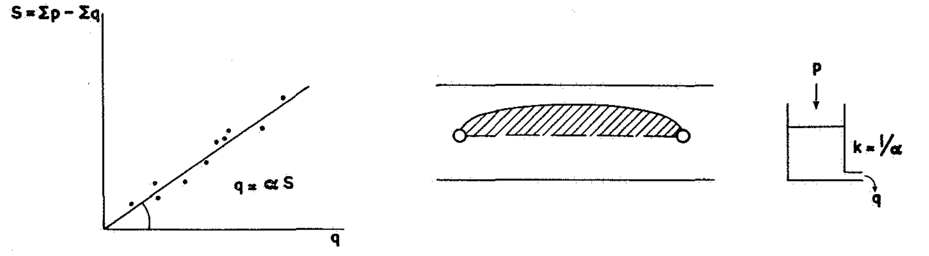


FIG. 15. Quasi-steady model of DE ZEEUW and HELLINGA (1952, 1958).

the ground water stored below the groundwater table between the parallel tile drains or ditches. The mathematical expression for the ratio between outflow rate and storage was found from a combination of the continuity equation and the steady-state relationship between the rate of flow to parallel drains and the storage below a groundwater table of elliptic shape (fig. 15).

$$q = \alpha S$$

$$\text{and } \alpha = 10 \frac{KD}{\mu L^2} \tag{30}$$

This is the expression for a single linear storage with a proportionality factor $k = 1/\alpha$. Consequently the lag of this model is $1/\alpha$ and the *IUH* can be expressed by

$$u(o,t) = \alpha e^{-\alpha t}$$

By its very nature this quasi-steady solution is suited to describe relatively slow variations of flow.

In his most recent models for natural drainage basins DE ZEEUW (in print) sometimes uses two or three parallel linear storages, whereas in other cases he places KRAIJENHOFF's model parallel to one or two linear storages. The contributions from these parallel storages to the total outflow are functions of the flow rate from the biggest storage which represents groundwater flow from higher grounds. Here a non-linear element of feed-back is introduced and consequently neither an *IUH* nor a constant time lag can be indicated. DE ZEEUW and HELLINGA were the first to use one compound hydrologic factor to typify a drainage situation. This factor expressed in (30) was based on the shapes of the groundwater table that were observed in the field by KIRKHAM and DE ZEEUW (1952).

WESSELING (1959) and VAN EYDEN (1959) developed quasi-steady routing methods based on the theoretical steady-state relation between groundwater storage and outflow. Their solutions obtained from a linearized equation for permanent flow to parallel drains are basically similar to those obtained by DE ZEEUW and HELLINGA but they find the proportionality factor:

$$k = \frac{1}{12} \frac{\mu L^2}{KD} \tag{31}$$

When these writers start from the full non-linear equation for permanent flow they arrive at complicated non-linear routing equations. These and other formulas for groundwater flow have been discussed by VAN KREGTEN in his excellent compendium (1963).

WEMELSFELDER (1963) studied the persistence of Rhine discharges on a basis of monthly flows. He neglected any carry-over of direct runoff from one month to the other and he assumed that only the slow component of groundwater flow is subject to the effect of linear storage. WEMELSFELDER further assumed that a certain fraction of direct runoff will replenish the

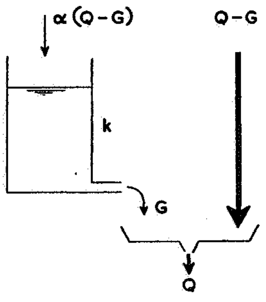


FIG. 16. Model for the relationship between river base flow and direct runoff( WEMELSFELDER, 1963).

ground water storage. This enabled him to calculate the "safe yield" of groundwater $G$ from the successive values of monthly total discharge $Q$. Figure 16 shows the model in an adjusted notation. Using this notation the "routing equation" can be written:

$$G_{t+1} = G_t e^{-1/l} + \frac{\alpha}{1+\alpha} Q_t (1 - e^{-1/l}) \qquad \text{where } l = \frac{k}{1+\alpha}$$

WEMELSFELDER varied the distribution factor $\alpha$ depending on the rate of total discharge $Q$, which implies the introduction of a non-linear element into this model.

The starting point for rainfall/runoff models in this country is the use of mathematical expressions for simplified non-steady groundwater flow processes. In many practical cases, however, the actual situation is far removed from these simplified basic models. In such cases many new factors interfere, rendering the model unfit for a sufficiently accurate description of the runoff process. In this situation new components are added to the model as a result of a more or less subjective evaluation of these various hydrological factors and then the initial advantage of a physically based runoff model rapidly fades away.

It is interesting to note that models developed in the sphere of physical hydrology appear to have a certain structural similarity with the models that arise from the field of linear system investigations where a good fit is the sole objective and the only physical argument may be that the runoff process is a heavily damped system (DOOGE, 1965).

In recent years a number of methods have been presented that eliminate subjective judgement to a high degree. They provide analytical ways to derive the *IUH* from the distribution in time of rainfall excess on one hand and the hydrograph of runoff on the other. In fact these methods perform analytically what is done graphically in the unit hydrograph method, but then they are free from subjective averaging procedures and the system response can be derived from any individual event of rainfall excess and its subsequent discharge.

O'DONNELL (1960) applied Fourier methods and DOOGE (1965) Laguerre functions to analyse input and output of a time invariant linear system and to define the impulse response. These methods will be extremely useful for the reconstruction of discharge hydrographs out of rainfall excess and thus for the determination of flood frequencies, but they leave the underlying physics of the runoff process out of consideration and therefore can give no insight into the changes in the regime to be expected from human interference. Since this is a major aspect of drainage problems in the Netherlands, physical hydrology will loose nothing of its importance along with the rapid development of new methods in systems analysis. The search for better physical models will have to continue.

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# III. METHODS OF COMPUTATION IN HYDROGRAPH ANALYSIS AND SYNTHESIS

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## 1. Introduction

The hydrograph of total streamflow at the outlet from a catchment is the overall continuing response of that catchment to the whole history of precipitation on the catchment. It has proved very beneficial in recent years to recognise that catchment behaviour is but one case of a general set of similar problems. This set is characterised by the presence of a system of some sort from which an output occurs as a response of the system to an input. "Systems engineering" has made remarkable progress in the last decade or so but hydrologists have been slow to adopt the powerful techniques made available by that progress. It is the purpose of this paper to present an account of the application to streamflow analysis and synthesis of certain computational techniques developed in systems engineering.

The treatment given, by no means a comprehensive one, describes some of the techniques available. In outline, the paper falls into two parts corresponding to the two broad classes into which systems can be divided: linear systems and non-linear systems (fig. 1). The first part of the paper is mostly concerned with those linear systems whose response is time-invariant, but also includes a brief mention of time-variant linear systems. The second part deals with non-linear systems. In both parts the techniques described are divided into (a) those in which the input to and output from a system can be treated by methods of analysis to yield information on the response characteristics of the system, and (b) those in which synthesis or simulation techniques that in effect provide mathematical models of catchment behaviour are used.

For the sake of completeness, the main body of the paper is preceded by a brief discussion of what constitutes a linear or a non-linear system. The author begs the indulgence of those to whom this discussion is superfluous and hopes that others will welcome an initial clarification of the division that is basic to the rest of the paper.

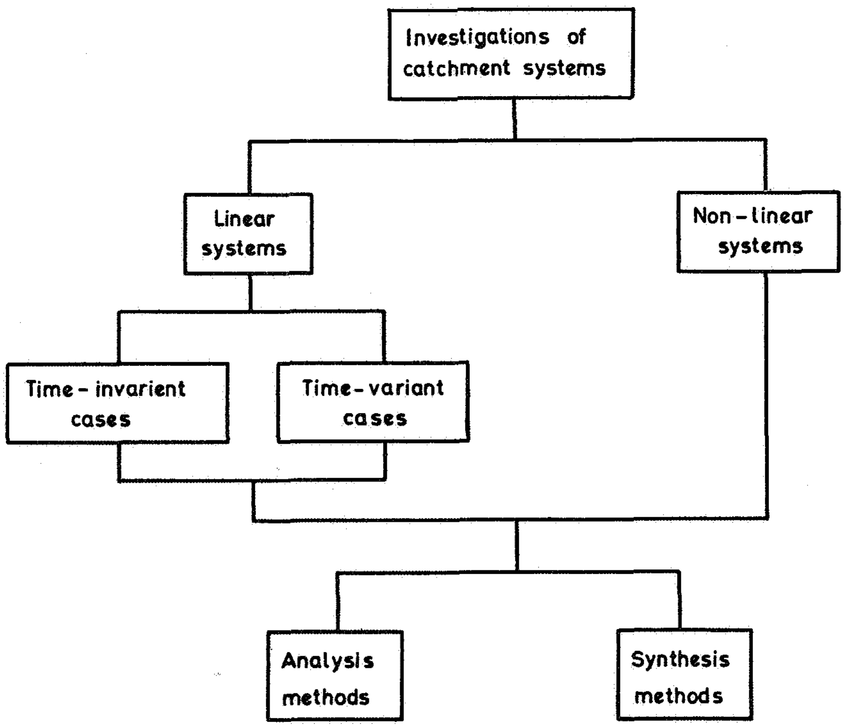


FIG. 1. Catchment studies via systems engineering methods.

## 2. LINEAR AND NON-LINEAR SYSTEMS

A system is said to be *linear* if its behaviour can be described by a linear differential equation. If $x(t)$ represents (as a fuction of time $t$) an input to a system and $y(t)$ the corresponding output from that system, then the system is linear if $y(t)$ is related to $x(t)$ by an equation* of the form

$$A_n \cdot \frac{d^n y}{dt^n} + A_{n-1} \cdot \frac{d^{n-1} y}{dt^{n-1}} + \ldots\ldots A_1 . \frac{dy}{dt} + A_0 y = x \qquad (1)$$

If $y$ or any differential of $y$ appeared other than to the first power, the system described by such an equation would be *non-linear.*

* Equation (1) is not the most general form of a linear differential equation and hence does not represent a general linear system. The use of a single equation restricts the system to a single input/single output system. The use of an ordinary differential equation means that the input and output are concentrated at definite points. The right-hand side of the equation will contain derivatives of $x$ if there are return loops in the system. The discussion here, however, will be confined to linear systems described by an equation of the form (1).

The coefficients $A_i$ ($i = 0,1 \ldots . n$) may be constants or may themselves be functions of $t$. A *time-invariant* linear system is one described by an equation of the form (1) in which all the $A_i$ coefficients are constants; if any or all coefficients are functions of $t$, the system described would be a *time-variant* linear system.

The outstanding property of all linear systems is that they obey the *principle of superposition.* If the output $y_1$ from a linear system described by an equation of the form (1) is caused by an input $x_1$ (i.e. $y = y_1$ is the solution of equation (1) when $x = x_1$), and $y_2$ is another output from the system due to another input $x_2$, then the output $y_3$ from an input $(x_1 + x_2)$ can be seen to be given simply by $y_3 = y_1 + y_2$. Also, if $x_2 = A.x_1$ then $y_2 = A.y_1$ (this proportionality property is a particular case of the additive property).

Such direct arithmetic superposition would not be permissible for a non-linear system. Put another way, in a linear system $y_2$ would be determined entirely by $x_2$ and would be independent of $x_1$; in a non-linear system $y_2$ would depend on $x_1$ as well as $x_2$.

For a time-invariant linear system, the output $y_1$ would always be the same whenever $x_1$ was applied. If the system were time-variant, however, then $y_1$ would depend on the time at which $x_1$ started as well as on $x_1$ itself. Generally speaking, non-linear systems are also time dependent with respect to start of input, but one must be careful to avoid the converse: a system which exhibits time dependence with regard to start of an input is not necessarily non-linear.

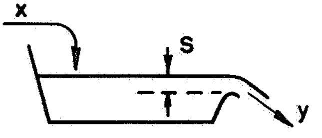


FIG. 2. A simple single-reservoir system.

To exemplify these concepts in a simple way, consider a system as in figure 2, consisting of a single reservoir for which the storage, $S$, is related to the output, $y$, by the relationship:

$$S = K.y^n \tag{2}$$

Then for an input $x$, we can write:

$$\frac{dS}{dt} + y = x \tag{3}$$

If in equation (2) we put $n = 1$, $K =$ a constant, then $\frac{dS}{dt} = K\frac{dy}{dt}$; inserting into equation (3) we get:

$$K \frac{dy}{dt} + y = x \tag{4}$$

Thus for $K$ constant and $n = 1$, the reservoir is a time-invariant linear system whose behaviour is described by equation (4).

If now $n = 1$ but $K = K(t)$, a function of time, then we have:

$$\frac{dS}{dt} = K(t)\frac{dy}{dt} + y\,\frac{dK(t)}{dt}$$

and insertion into equation (3) now gives:

$$K(t)\,.\,\frac{dy}{dt} + \left[1 + \frac{dK(t)}{dt}\right].\,y = x \tag{5}$$

So for $K$ dependent on time and $n = 1$, we have a time-variant linear system whose behaviour is described by equation (5).

Finally, consider the case when $n \neq 1$ and, for simplicity, $K =$ a constant. Then $\frac{dS}{dt} = Kny^{n-1}\frac{dy}{dt}$ and the equation describing the system is:

$$Kny^{n-1}.\,\frac{dy}{dt} + y = x \tag{6}$$

i.e. for $n \neq 1$, whatever $K$, the reservoir is a non-linear system.

In conclusion, it requires only one non-linear component in a system of many components to make that system non-linear.

## 3. Linear treatments of catchment behaviour

Two of the basic assumptions of the *unit hydrograph method* of relating a rainfall excess on a catchment to the resulting hydrograph of direct runoff are:

(1) invariance of response — the same rainfall excess, whenever it is applied, will always produce the same direct runoff hydrograph;

(2) superposition of responses — the runoff due to two or more different rainfalls applied together is the arithmetic sum of the separate runoffs caused by each of the rainfalls applied separately.

Unit hydrograph theory is therefore based on the assumption that the catchment is a time-invariant linear system, at least so far as rainfall excess and direct runoff are concerned.

The T-hour unit hydrograph (*TUH*) of a catchment, written $u(T, t)$, may be defined as the direct runoff hydrograph due to unit volume of rainfall excess falling uniformly over the catchment in a period of $T$ hours. If we make $T$ smaller, keeping the volume of rainfall excess constant at unity, in the limit (as $T$ approaches zero) the *TUH* approaches the instantaneous unit hydrograph (*IUH*). In systems engineering terms, the *IUH* is the *impulse response* of the catchment system, i.e. the output from the system due to an instantaneous impulse input of unit size.

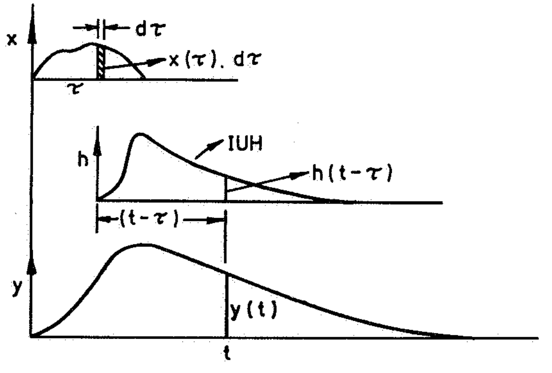


FIG. 3. The convolution operation.

The impulse response of a time-invariant linear system is the key characteristic of that system. From it the response due to any other input can be derived. If we write $h(t)$ for the *IUH* of a catchment, then the direct runoff output, $y(t)$, is related to its causative rainfall excess input, $x(t)$, analytically by equations (7) (known as the convolution operation) and diagrammatically by figure 3.

$$\left.\begin{aligned} y(t) &= \int_0^t x(\tau)\,.\,h\,(t-\tau)\,.\,d\tau \\ &= \int_0^t h(\tau)\,.\,x(t-\tau)\,.\,d\tau \\ &= h(t) * x(t) \end{aligned}\right\} \quad (7)$$

The first version of the convolution integral in equation (7) is the limiting case of the usual finite-period unit hydrograph procedure (illustrated in fig.

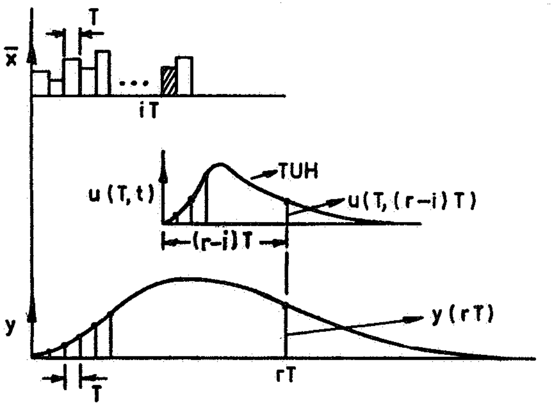


FIG. 4. The unit hydrograph procedure.

4) i.e. given a block rainfall histogram of mean intensities over intervals $T$ $(\bar{x}_0, \bar{x}_1, \bar{x}_2 \ldots . \bar{x}_i \ldots .)$

$$\text{then } y(rT) = T \sum_{i=0}^{r} \bar{x}_i . \, u\,(T, (r-i)T) \tag{7a}$$

In the limit, as $T \rightarrow 0$, the summation of (7a) is replaced by the integral in (7).

An interesting case using the second version of the convolution integral in (7) is to put $x(t-\tau) = \frac{1}{T}$ for $0 \leqslant (t-\tau) \leqslant T$ but zero outside this range. This implies a unit volume of rainfall excess occurring uniformly over a period $T$ i.e. the runoff will be, by definition, the *TUH*. For this case, the range of $\tau$ is effectively from $(t-T)$ to $t$ ($x(t-\tau)$ being zero otherwise) so these limits appear on the integral. We then get equation (8) which relates ordinates of the *TUH* to partial areas of the *IUH*:

$$y(t) = u(T,t) = \frac{1}{T}\int_{t-T}^{t} h(\tau) \, . \, d\tau \tag{8}$$

(with the lower limit replaced by zero if $t < T$).

So, given the *IUH*, any design storm having specified volumes of rainfall excess occurring in intervals of length $T$ can be found by first finding the *TUH* and then performing the usual finite-period unit hydrograph summation of equation (7a). This can be done for any desired value of $T$, thus permitting storm data from intensity/duration/frequency curves to be examined for a "worst" case.

Clearly, then, the *IUH* is of great interest and the question of how to find an *IUH* is an important one. It will be appreciated that equation (7) can readily be used to find the output, $y(t)$, from a given input, $x(t)$, and a known impulse response, $h(t)$. However, the problem of finding $h(t)$ from given $y(t)$ and $x(t)$ is not straightforward. Earlier work on this problem invoked the use of linear catchment models which, when fitted to catchment data, were then taken to have an impulse response equal to the *IUH* of the catchment. This indirect *synthesis* approach has been followed by more direct *analysis* methods which endeavour to yield the *IUH* directly from catchment data without postulating any specific model mechanism.

It is relevant to emphasise again that unit hydrograph theory implies a linear mechanism for the transformation of rainfall excess into surface run-off. Should evidence be produced that the groundwater system also discharges baseflow into the stream via a linear transformation of percolation, then techniques of analysis of linear systems would be applicable to the whole of streamflow (making due allowance for "loss" of part of the total input via evapotranspiration). The fact that such total streamflow may never drop completely to zero merely implies that parts of the linear system have extremely long "memory" times. Appropriate steps can be taken in applying linear analysis methods to such long memory systems. The simplest might be to terminate the hydrograph after a drought period by some simple objective criterion on magnitude of discharge, i.e. assume that the total streamflow, when it is suitably small, has become zero.

### *a. Linear system synthesis*

#### *a. 1. Time-invariant linear models*

#### The Nash cascade model

Nash (1960) made a most significant contribution to unit hydrograph theory by showing that although it might be difficult to find the precise *shape* of the *IUH* it is a very simple matter to derive certain overall properties of that shape, viz. its moments of area (fig. 5).

For a time-invariant linear system described by an equation of the form (1), Nash showed that the first moments (about the *origin*) of the impulse response of that system, any input to the system and the corresponding output from the system ($H_1'$, $X_1'$ and $Y_1'$ respectively) are related by:

$$H_1' = Y_1' - X_1' \qquad (9)$$

and that the second moments (about the *centres of area*), $H_2$, $X_2$ and $Y_2$, are related by

$$H_2 = Y_2 - X_2 \qquad (10)$$

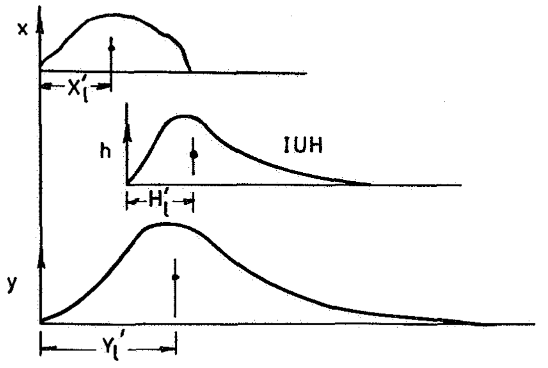


Fig. 5. Moments of area for a linear system.

Similar relations, getting somewhat more complex, exist for the higher moments.

These moment relationships, stemming purely and simply from the assumption that the catchment behaves as a time-invariant linear system, involve no approximation whatsoever — they are fundamental proporties af any such system.

It is a simple matter to calculate the moments of area of rainfall and runoff curves, so the moments of an *IUH* for a given catchment can be found simply and directly from storm data recorded on that catchment via moment equations such as (9) and (10).

In order to complete the search for the *IUH* i.e. estimate its shape, Nash set out to develop a sufficiently general linear model of catchment behaviour. Any such model has an impulse response for which some general analytical equation, expressed in terms of the parameters of the model, can be derived. From this equation, expressions for the moments of area of the model impulse response can also be derived, again in terms of the parameters of the postulated model.

For any given actual catchment data, numerical values of the *IUH* moments got from equations such as (9) and (10) when equated to the general expressions for the moments of the model impulse response will yield numerical estimates of the model parameters. These estimates, if substituted back into the general analytical equation for the model impulse response, will give a specific version of that equation. This is taken to be an approximation to the *IUH* of the catchment in question.

In other words, a general model of a class of linear systems is fitted to any specific member of that class by matching the moments of area of the general model impulse response to those of the specific system impulse response (got directly from a set of input and output data for that system). Clearly, though there is no approximation (in principle) in finding the moments of the impulse response of the real system (inevitably, data errors will enter into the matter), there is an approximation implicit in the matching process: the model response can only be as good a representation of a real system response as the model itself is a good representation of reality.

After considerable investigation, NASH (1960) chose as a sufficiently adequate general model of linear catchments a cascade of $n$ identical linear reservoirs in series, all having the storage characteristic:

$$S = K.\, y \tag{11}$$

This two-parameter model calls for only two moment equations in order to solve for the values of the parameters $n$ and $K$ when matching any given catchment.

The impulse response of the NASH-model is:

$$h(t) = \frac{1}{K.\, \Gamma(n)} \left(\frac{t}{k}\right)^{n-1} e^{-t/k} \tag{12}$$

in which $\Gamma(n)$ is the gamma function and is tabulated ($\Gamma(n) = \Gamma(n-1).(n-1)$ and, if $n$ is an integer, $\Gamma(n) = (n-1)!$). For equation (12), the first moment (about the origin) is $nK$ while the second moment (about the centre of area) is $nK^2$, so the process of finding $n$ and $K$ values from catchment data via equations (9) and (10) is extremely simple:

$$n = \frac{(Y_1' - X_1')^2}{Y_2 - X_2} \tag{9a}$$

$$K = \frac{Y_2 - X_2}{Y_1' - X_1'} \tag{10a}$$

Once specific parameter values have been found for equation (12) for any given catchment, equation (8) calls for integration of the *IUH* to yield any *TUH*. As NASH remarks, this can be done with tables of the incomplete gamma function, but of course numerical integration would be just as satisfactory and would be the natural choice for any computer usage of the NASH-technique. In fact just the tedious repetitiveness of moment calculations on catchment data alone demands a digital computer solution of the NASH-technique. Such a solution might follow the general flow diagram shown in figure 6.

A second and highly practical topic treated extensively by NASH (1960) is the question of estimating the *IUH* (and *TUH*'s) of an *ungauged* catchment. Taking various physical features (area, slope, channel length etc.) of 90 gauged catchments in the United Kingdom, NASH established good correla-

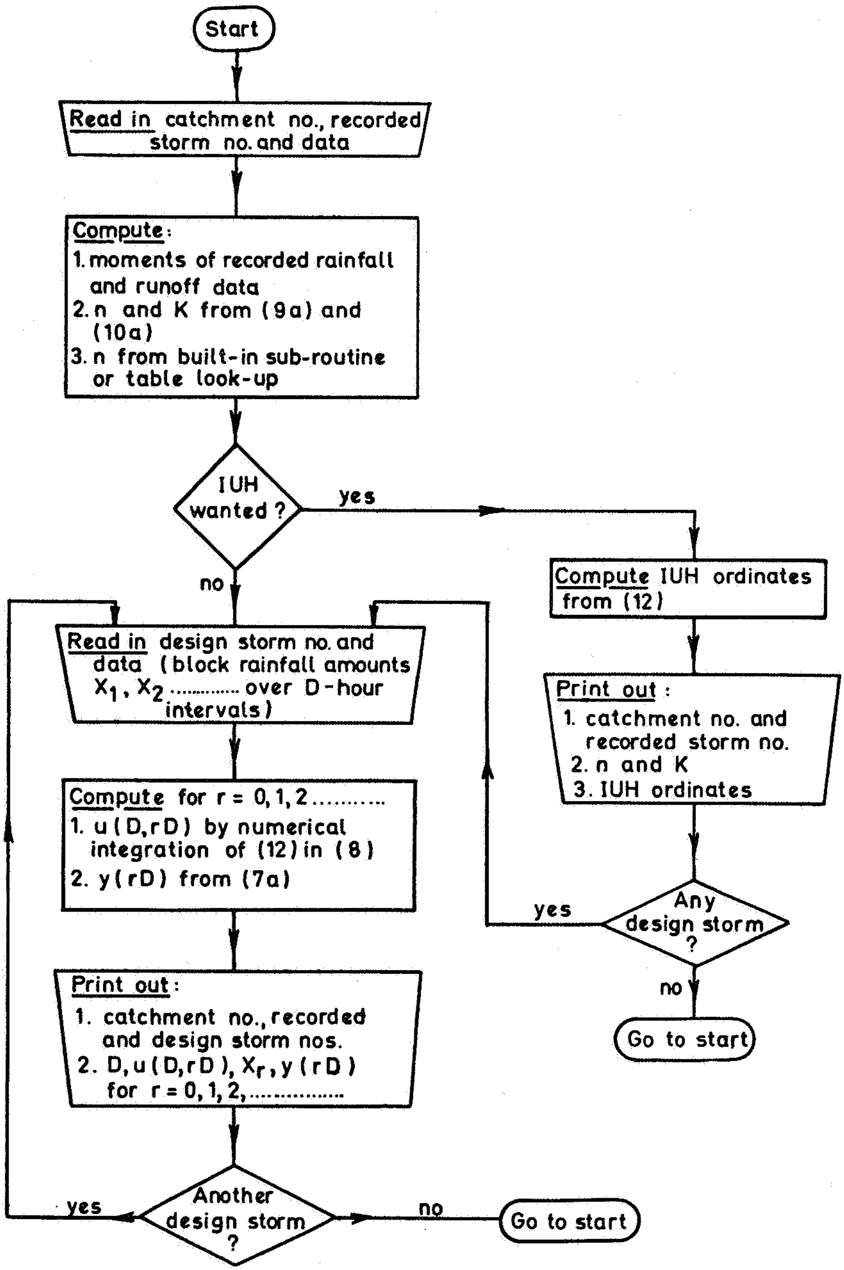


FIG. 6. Flow diagram for Nash technique.

tions between these factors and dimensionless groupings of the various *IUH* moments (got directly from the measured rainfall and runoff data via the moment relationships). For any ungauged catchment, one would enter the correlation equations with values of the relevant physical factors for the catchment and so derive numerical estimates of its *IUH* moments. These would yield the $n$ and $K$ values necessary to carry out the required evaluation of the *IUH* (and any *TUH*).

## The Dooge model

One of the criticisms that can be made of the NASH-model is that it allows for reservoir storage effects but not for channel translation effects present in any catchment. In his brilliant fundamental study of the general theoretical basis of the unit hydrograph method, DOOGE (1959) developed a model technique which allows for translatory as well as storage effects in catchments.

In brief, DOOGE's general analysis of linear catchment systems postulates that the output from an element of a catchment incurs a translation delay time, $\tau$, and also passes through $n$ linear reservoirs in its passage to the catchment outlet, $n$ being dependent on $\tau$. The resulting rather unwieldy general equation for the impulse response of such a system is greatly simplified (without, as DOOGE shows, appreciable effects on the results) by two assumptions: (a) all elements having the same $\tau$ value have the same chain of linear storages to the outlet and (b) all the storages have the same storage characteristic, $K$. These assumptions yield a model system that can be represented diagrammatically as in figure 7.

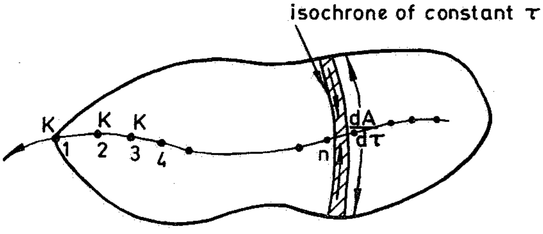


FIG. 7. The Dooge general catchment model.

Bearing in mind that both $n$ and $\frac{dA}{d\tau}$ are functions of $\tau$, DOOGE's final equation for the impulse response of this model to a non-uniformly distributed input can be written:

$$h(t) = \frac{1}{A} \int_{0}^{t/K} \left[ \frac{m^{n-1}\, e^{-m}}{(n-1)!} \right] \cdot \left( i \frac{dA}{d\tau} \right)_{\tau} \cdot dm \qquad (13)$$

in which $m = \frac{t-\tau}{K}$, $\left(\frac{dA}{d\tau}\right)_\tau$ is the ordinate of the time-area curve at time $\tau$ and $i$ the average rainfall intensity along each isochrone (varying only with distance from the outlet).

A procedure for evaluating equation (13) is presented by DOOGE (1959). The quantity in square brackets (the Poisson probability function) is available in tables and so the integral can be evaluated numerically for any given $t$ value by taking increments of $\tau$ from 0 to $t$. The distribution of linear reservoirs (as a function of $\tau$) must be known or assumed and the time-area curve has to be found (again as a function of $\tau$) in order to carry out the integration. The $n(\tau)$ distribution may be found by a moments technique outlined by DOOGE (1959) or, possibly, by a Laguerre function analysis, similar to the one described in section 3.b.2.

### *a. 2. Time-variant linear models*

All the published work known to the author on linear systems analysis of catchment behaviour has followed the classical unit hydrograph approach with regard to invariance of response. The linear catchment mechanism has always been taken to be time-invariant. Much encouraging work, some of it described in section 4, has been reported by hydrologists working with non-linear techniques of synthesis and analysis, usually time-variant. There would seem to be room for study of the missing area — time-variant linear systems — before abandoning the very powerful principle of superposition, as is necessitated by non-linear studies. In particular, those linear systems in which the time variance is of a cyclical or seasonal nature might well prove worthy of study.

The author has made some tentative preliminary studies of one approach to this field of study. Though not yet very useful as a practical tool for hydrograph computations, a brief discussion of the approach may indicate a potential usefulness.

#### A time-variant cascade model

The starting point of these studies was to take the NASH-type model consisting of a cascade of equal linear storages and allow the storage characteristic, $K$, to become time-dependent. Thus for the $r^{th}$ reservoir in the cascade, the equation of continuity relating input, $x_r$, output, $y_r$, and rate of change of storage, $\frac{dS}{dt}$, is:

$$x_r = y_r + \frac{dS}{dt} \tag{14}$$

Now $x_r$ is the output from the $(r-1)^{th}$ reservoir, and, if $S = K.y_r$ where $K = K(t)$, equation (14) becomes:

$$(1 + \frac{dK}{dt}) .y_r + K\frac{dy_r}{dt} = y_{r-1}$$

or

$$\frac{dy_r}{dt} + P.y_r = \frac{1}{K}.y_{r-1} \tag{15}$$

in which $P = \frac{1}{K}(1 + \frac{dK}{dt})$.

Equation (15) has the integration factor $K.e^{\int \frac{dt}{K}}$. Putting $Q = e^{\int \frac{dt}{K}}$, integration of (15) yields:

$$y_r.K.Q = \int y_{r-1}.Q.dt + \text{a constant} \tag{16}$$

in which both $K$ and $Q$ are functions of $t$.

To evaluate the impulse response, $h(t)$, of any such system of $n$ linear time-variant storages to an instantaneous input of unit volume into the first storage, one must solve equation (16) recursively from $r = 1$ to $r = n$ when, by definition, $h(t) = y_n$. To do this one must know how $K$ depends on $t$.

For simple cases, a solution by formal analysis is possible. For example, if $K = K_o + at$, in which $K_o$ is a given initial value of $K$ at $t = 0$, then $Q = K^{1/a}$ and formal recursive integration of (16) gives:

$$h(t) = y_n = \frac{1}{K_0.(n-1)!} \frac{\left\{ \log_e \left[1 + \frac{\alpha t}{K_0}\right]^{1/\alpha} \right\}^{n-1}}{\left[1 + \frac{\alpha t}{K_0}\right]^{1 + 1/\alpha}} \tag{17}$$

In this equation, $t$ is measured from the instant at which the unit impulse is applied, at which time $K = K_o$. In convoluting the impulse response of (17) with any arbitrary input $x$, one must make due allowance for the fact that $h(t)$ is different for impulses applied at different times. Defining $t = 0$ now as the start of the input $x$, with $K = K_o$ at $t = 0$, the element of input $x(\tau).d\tau$ applied at time $t = \tau$ (fig. 8) must be associated with the impulse response appropriate to that time $\tau$. The impulse response may now be written $h(\tau, \varepsilon)$ in which the running time, $\varepsilon$, is measured from a zero at the instant $\tau$ and for which $K_o$ in (17) must be replaced by $K_\tau = K_o + \alpha\tau$ i.e.

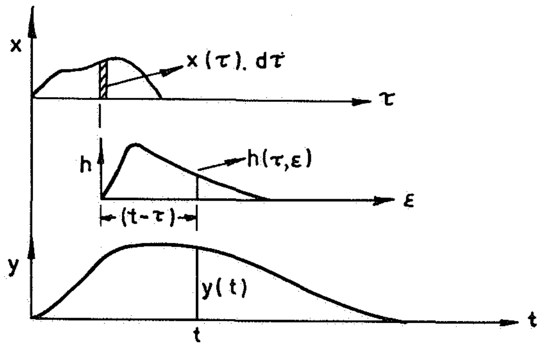


FIG. 8. Time-variant linear convolution .

$$h(\tau,\varepsilon) = \frac{1}{K_\tau . (n-1)!} . \frac{\left\{ \log_e \left[ 1 + \frac{\alpha\varepsilon}{K_\tau} \right]^{1/\alpha} \right\}^{n-1}}{\left[ 1 + \frac{\alpha\varepsilon}{K_t} \right]^{1+1/\alpha}} \tag{18}$$

The output $y$ at time $t$ is now given by:

$$y(t) = \int_0^t x(\tau).h(\tau,t-\tau).d\tau \tag{19}$$

Using (18), this eventually reduces to:

$$y(t) = \frac{1}{K_t^{\,1+1/\alpha} . (n-1)!} \int_0^t x(\tau) \frac{\left\{ \log_e \left[ \frac{K_t}{K_\tau} \right]^{1/\alpha} \right\}^{n-1}}{K_\tau^{1/\alpha}} d\tau \tag{20}$$

(in which $K_t = K_o + \alpha t$).

The simple moment relationships for the input, output and impulse response of a time-invariant linear system (equation (9) and (10)) do not hold for a time-variant system, so the question arises as to how one would set about finding the parameters of a time-variant model from records of input and output data. One possible method might be the optimisation technique described in section 4.a.2. Such a technique would start with some initial estimates of the parameters of a model (e.g. $n$, $K_o$ and $\alpha$ of the above case) and automatically adjust them to yield a "best-fit" set.

The simplest possible case of time-variant storage in only one class of model has been presented above, largely as an illustration of a possible line of enquiry. Even with this single cascade model, more complex parametric equations for cyclical or seasonal variation of $K$ with time generally prohibit formal solution of equation (16) to find $h(t)$. However, once numerical values are allotted to a set of model parameters, one can always switch to numerical integration techniques to solve for $Q$ and then $y_r$ recursively in (16) in order to evaluate $h(t)$ numerically. Convolution can also be performed by numerical integration. Optimisation of the model parameters could thus be carried out over the whole process, not just from a formal result like equation (20). This approach could, of course, be applied to models other than the single cascade model having equal storages considered above.

*b. Linear system analysis*

To repeat an earlier comment, deriving a catchment *IUH* by invoking the aid of some catchment model is bound to lead to an approximation to the real *IUH*, the goodness of the approximation being dictated by the goodness of the simulation by the model of reality. One would like to use methods of deriving an *IUH* or a *TUH* that by-pass the need for a model, i.e. operate directly on the rainfall excess and direct runoff data to yield the *IUH* or *TUH*. Such methods have, in fact, been used and require only that the system be linear and time-invariant.

One such method involves the use of *orthogonal functions* to derive an *IUH* or a *TUH;* another uses *matrix techniques* to derive a *TUH*. Other techniques of analysis of linear systems (time-variant as well as time-invariant) are available and either have been or are waiting to be explored, Laplace transform methods (Diskin, 1964) being one such technique. The discussion here will be limited, however, to two orthogonal function techniques and one matrix technique.

In general a function $g(t)$ can, for all values of $t$, be represented exactly by the sum of an infinite series of other functions:

$$g(t) = \sum_{m=0}^{\infty} c_m \cdot f_m(t) \tag{21}$$

in which the $c_m$ coefficients are constants. Examples are the polynomial series:

$$g(t) = c_o + c_1 t + c_2 t^2 + \ldots .$$

and the Fourier series:

$$g(t) = a_o + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots.$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots.$$

Amongst many such series, there are some in which the $f_m(t)$ functions have the property:

$$\left.\int_a^b f_m(t).f_n(t).\,dt = 0 \text{ if m} \neq n \atop \text{but} = K \text{ if } m = n \right\} \qquad (22)$$

Such functions are called *orthogonal functions*.* The limits on the integral in (22) and the value of the constant, K, depend on the particular class of orthogonal function being used. (If K = 1, the functions are then said to be *orthonormal*).

The great value of orthogonal functions lies in the ease with which the coefficients $c_m$ in equation (21) can be obtained. If we multiply both sides of (21) by $f_m(t)$, integrate between the limits $a$ and $b$ and make use of (22), we get:

$$\int_a^b g(t).\,f_m(t).\,dt = 0 + 0 + \ldots. + c_m.K + 0 + 0 + \ldots.$$

$$\text{i.e. } c_m = \frac{1}{K}\int_a^b g(t).\,f_m(t).\,dt \qquad (23)$$

Turning now to the time-invariant linear system problem, let us write the input, $x(t)$, the output, $y(t)$, and the impulse response $h(t)$ as follows:

$$\left.\begin{aligned} x(t) &= \sum_o^\infty (c_x)_m.\, f_m(t) \\ y(t) &= \sum_o^\infty (c_y)_m.\, f_m(t) \\ h(t) &= \sum_o^\infty (c_h)_m.\, f_m(t) \end{aligned}\right\} \qquad (24)$$

* Equation (22) differs slightly from the form usually presented in that a weighting factor normally present inside the integral has been assumed to be absorbed into the functions themselves. In addition, (22) defines orthogonality with respect to *integration*: there is a separate but corresponding definition with regard to *summation*.

in which the same orthogonal functions, $f_m(t)$, are used for all three expansions (and, since the system is time-invariant, all the coefficients are constants).

Our problem is, knowing $x(t)$ and $y(t)$, how do we find $h(t)$? Put another way, if we know the coefficients $(c_x)_m$ and $(c_y)_m$ — and we do, from equation (23), if we know $x(t)$ and $y(t)$ — can we find the coefficients $(c_h)_m$ (and hence the impulse response, $h(t)$)?

The obvious line of attack is to make use of the fundamental relationship of any linear system, viz. the convolution integral of equation (7):

$$y(t) = \int_o^t x(\tau).\, h\,(t\text{-}\tau).\, d\tau \qquad (7 \text{ bis})$$

We can hope that insertion of equations (24) into equation (7) will yield useful simple algebraic relationships between the known $(c_x)_m$ and $(c_y)_m$ and the unknown $(c_h)_m$.

Two specific types of orthogonal fuctions have been examined for this purpose and have yielded useful results.

*b. 1. Inversion via Fourier series*

A function $g(t)$ that exists within the range $0 \leqslant t \leqslant L$ can be represented exactly at every point in that range by the infinite Fourier series:

$$g(t) = a_o + \sum_{r=1}^{\infty} \left\{ a_r.\cos r \frac{2\pi t}{L} + b_r.\, \sin r \frac{2\pi t}{L} \right\} \qquad (25)$$

The cosine and sine functions of this series are orthogonal to one another for any pair of limits $L$ apart, yielding a $K$ value (equation (22)) equal to $L/2$ Thus the coefficients in equation (25) are given by:

$$\left.\begin{aligned} a_r &= \frac{2}{L}\int_o^L g(t).\cos r \frac{2\pi t}{L}.dt \quad \text{but } a_o = \frac{1}{L}\int_o^T g(t).dt \\ b_r &= \frac{2}{L}\int_o^L g(t).\sin r \frac{2\pi t}{L}.dt \end{aligned}\right\} \qquad (26)$$

Referring back to figure 1, consider a time-base $L$ greater than or equal to the length of the direct runoff hydrograph, $y(t)$. Let the three functions

$x(t)$, $y(t)$ and $h(t)$ be represented by Fourier expansions of the form of (25), all over the same time-base $L$. Using coefficients $[a, b]$ for $x(t)$, $[A, B]$ for $y(t)$ and $[\alpha, \beta]$ for $h(t)$, O'DONNELL (1960) has shown that substitution into the convolution integral (equation (7)) yields the following simple algebraic relationships between the coefficients:

$$\left.\begin{aligned} A_n &= \frac{L}{2}(a_n\alpha_n - b_n\beta_n) \qquad \text{but } A_o = L.a_o\alpha_o \\ B_n &= \frac{L}{2}(a_n\beta_n + b_n\alpha_n) \end{aligned}\right\} \qquad (27)$$

Solving for $\alpha_n$ and $\beta_n$:

$$\left.\begin{aligned} \alpha_n &= \frac{2}{L}\,\frac{a_nA_n + b_nB_n}{a_n^2 + b_n^2} \qquad \text{but } \alpha_o = \frac{1}{L}\,\frac{A_o}{a_o} \\ \beta_n &= \frac{2}{L}\,\frac{a_nB_n - b_nA_n}{a_n^2 + b_n^2} \end{aligned}\right\} \qquad (28)$$

In principle, then, given an $x(t)$ and the corresponding $y(t)$ for a time-invariant linear system, one can use equations (28) to find the Fourier coefficients of the impulse response of that system from the coefficients derived from the $x(t)$ and $y(t)$ via equations (26). One can then either synthesise $h(t)$ itself from its coefficients (via an equation of the form (25)) or use these coefficients with those of a new (design) input in equations (27) to derive the coefficients of the corresponding new output. From the latter, the output itself would be synthesised.

In practice, two restrictions arise when applying the Fourier technique to the catchment problem. As with any experimentally observed system, the input and output can only be *sampled*, i.e. they can be measured at only a finite number of observation points (even a continuous chart record can be read only at discrete points). Furthermore, the input data in the catchment problem is usually given as a block histogram (volumes in equal periods of time) not as a continuous *intensity* function (the output data usually does consist of spot intensity values read from a continuous chart record or recorded at discrete instants of time).

When we know only a finite number of (equally spaced) ordinates on a continuous but *unknown* function $g(t)$, we cannot perform analytically the integrations of equations (26) to find the Fourier coefficients ($g(t)$ is not known) nor can we expect to establish the *infinite* series of equation (25) to define everywhere the (unknown) $g(t)$. However we can fit the known ordinates with a *finite* trigonometric series having a form similar to (25) and, in order to find the coefficients of this finite series, we can replace the integrals of (26) by sums.

Let the known $g(t)$ be sampled at $n$ equidistant data points in the range $0 \leqslant t \leqslant L$ i.e. we know

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $g(0)$ | $g(h)$ | $g(2h)$ | .... | $g(ih)$ | .... | $g([n-1]h)$ |
| at | $t = 0$ | $h$ | $2h$ | .... | $ih$ | .... | $[n-1]h$ |

Further let $n$ be odd, $= 2p + 1$ say, and let $L = nh$ (it is convenient, though not neccessary, to make $g(0) = g(L) = 0$).

Then we can fit the $n$ data points exactly by the finite *harmonic* series of $n$ terms:

$$g(ih) = a_o + \sum_{r=1}^{p} (a_r \cos r \frac{2\pi i}{n} + b_r . \sin r \frac{2\pi i}{n}) \tag{29}$$

(This is, in effect, a set of $n$ simultaneous equations (for $i = 0, 1, 2 \ldots\ldots (n-1)$) with $n$ "unknown" coefficients $a_0, a_1 \ldots\ldots a_p; b_1, b_2 \ldots\ldots b_p$.) The orthogonality property, this time with respect to summation, permits us to find the $n$ coefficients:

$$\left.\begin{aligned} a_r &= \frac{2}{n} \sum_{i=o}^{n-1} g(ih) \, . \cos r \frac{2\pi i}{n} \qquad \text{but } a_o = \frac{1}{n} \sum_{i=o}^{n-1} g(ih) \\ b_r &= \frac{2}{n} \sum_{i=o}^{n-1} g(ih) . \sin r \frac{2\pi i}{n} \end{aligned}\right\} \tag{30}$$

The series (29) can hardly be expected to fit $g(t)$ exactly *between* the given data points where in fact $g(t)$ is not known but it does fit the $n$ data points exactly. The $n$ *harmonic* coefficients of the finite series fitting $n$ points on a continuous function will, as $n$ is increased, approach the Fourier coefficients of the finite series fitting the function everywhere.

The implications of this sampling restriction are twofold. We can use only harmonic coefficients as approximations for the Fourier coefficients $(a_n, b_n)$ and $(A_n, B_n)$, in equations (28) and we can find only a finite number of such approximate Fourier coefficients. We must therefore accept errors in the $\alpha_n$, $\beta_n$ coefficients of the *IUH* response and be limited to a finite number of such coefficients. Errors will therefore appear in the synthesis of the *IUH* ordinates due both to the approximations in the coefficients and to the fact that we can find only a finite number of them.

However, when we turn to consider the second restriction (that the input data for catchments usually comes as a histogram) some measure of relief from the sampling restriction is possible. At first sight it would seem unpromising that instead of a continuous *intensity* curve of input (sampled every $h$ units of time) we have available only the *volumes* of this input integrated over successive time periods of length $h$. But in the absence of any other knowledge of the true input intensity curve, the only proper assumptions we should make are that the input truly does occur in a series of constant intensities, $\bar{x}_i$, over those time intervals, and that the recorded output is the true result of such an input. This supposition immediately frees us from having to use harmonic coefficients for the input since we can now make use of equations (26) (with $g(t) = \bar{x}_i$ a constant) for $ih \leqslant t \leqslant (i + 1)h)$ to find as many true Fourier coefficients of the histogram of input as we like. In addition, we are now free to read off from a continuous chart, if available, many more output data points within the intervals of length $h$ and do not have to restrict ourselves to fewer points spaced $h$ apart. This will yield harmonic coefficients which are much better estimates of the Fourier coefficients of the output and so improve the definition of the *IUH*.

It is not essential that $n$ should be odd, as defined in the above discussion, but there are some slight advantages in manipulation and programming if $n$ is kept odd. Of greater usefulness is the deliberate addition of "extra data" in the form of zeros at the end of the real (non-zero) runoff data. By choosing $n$ in equations (29) and (30) such that $L$ is, say, 25% greater than the actual time basis of the runoff hydrograph, and so ending the sequence of runoff ordinates with a number of zeros, some spurious oscillations near the tail of the computed *IUH* due to truncation of the infinite series are displaced away from that region to later ordinates. The latter, it is known, should be zero so the oscillations can be ignored in that region.

In concluding this account of the Fourier technique of deriving an *IUH*, it is of interest to note that the same form of manipulation may be used to find a *TUH* directly and exactly without first finding the relevant *IUH*. Consider again the usual form of presentation of the unit hydrograph method as given in figure 4. $y(rT)$ is, as before, taken to be the true resulting output due to an input given exactly by the histogram shown viz.

$$x(t) = \bar{x}_i \text{ for } iT \leqslant t \leqslant (i+1)\,T \quad i = 0, 1, 2 \ldots .$$

That is to say:

$$y(rT) = T \sum_{i=0}^{r} \bar{x}_i \,.\, u(T, \{r-i\}T) \qquad r = 0, 1, 2 \ldots (n-1) \qquad \text{(7a bis)}$$

and in which $u(T, kT)$ ($k = 0, 1, 2 \ldots (n-1)$) is the *TUH*.

Let us now define three finite harmonic series, each of $n$ terms (and having the common fundamental time base $L = nT$) viz.

(1) one for the input, having coefficients $[c, d]$, and fitting exactly $n$ data points consisting of the mean intensity values, $\bar{x}_i$, taken at the *start* of each interval;

(2) the second for the output, having coefficients $[C, D]$, and fitting exactly the ordinates $y(rT)$;

(3) the third for the *TUH* having coefficients $[\gamma, \delta]$ and fitting exactly the ordinates $u(T, kT)$.

(None of these series does more than pass through $n$ points spaced $T$ apart on its respective function; *between* the data points each series will generally differ from the true function on which those data points lie.)

Substitution of such series into equation (7a) leads eventually to the following relationships for the $[\gamma, \delta]$ coefficients of the *TUH*:

$$\left.\begin{aligned} \gamma_n &= \frac{2}{L} \cdot \frac{c_n C_n + d_n D_n}{c_n^2 + d_n^2} \qquad \text{but } \gamma_o = \frac{1}{L} \frac{C_o}{c_o} \\ \delta_n &= \frac{2}{L} \cdot \frac{c_n D_n - d_n C_n}{c_n^2 + d_n^2} \end{aligned}\right\} \qquad (31)$$

Equations (31) are of precisely the same form as equations (28). It will be seen, however, that there are now *no* approximations involved in evaluating the *harmonic* $[\gamma, \delta]$ coefficients from the *harmonic* $[c, d]$ and $[C, D]$ coefficients. Further, we are now no longer troubled by the Fourier series truncation problem since we can find exactly *all* the $n$ coefficients of the finite $n$-term harmonic series passing through $n$ equidistant points on the *TUH*, and these $n$ coefficients are sufficient to synthesise exactly the $n$ *TUH* ordinates.

It should be pointed out that the method just described does not give the *whole* of the *TUH;* it merely gives $n$ ordinates on the *TUH*, but gives them exactly (in principle). The method described earlier for deriving an *IUH* (using Fourier coefficients for $\bar{x}_i$) yields an *equation* for the *IUH*, one which may be

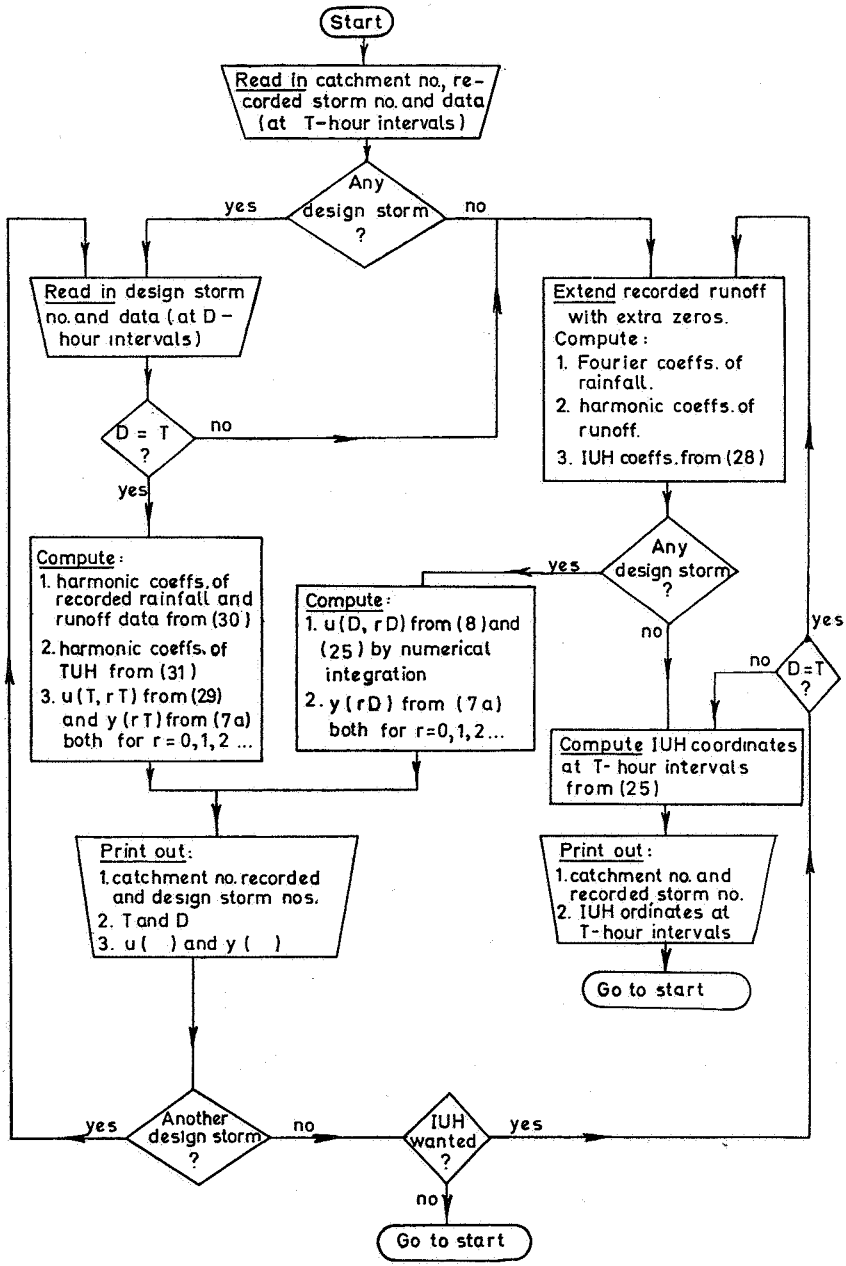


FIG. 9. Flow diagram for Fourier technique.

taken as an approximation to the *whole* of the *IUH*; if evaluated at $n$ equidistant intervals, the calculated *IUH* ordinates will differ from their true values due to the approximation inherent in finding the *IUH* Fourier coefficients from approximations to the $y(t)$ Fourier coefficients and also to the fact that we can find and use only a finite number of the approximate *IUH* Fourier coefficients.

It hardly needs saying that the Fourier technique would be impracticable to use without the aid of a digital computer. The calculations must be carried out with a large number of significant figures, not less than thirty or so terms in the various series are needed to achieve satisfactory results and most of the calculations are repetitive. All these features demand a computer solution along the lines indicated in the flow diagram of figure 9.

*b. 2. Inversion via Laguerre functions*

Another class of orthogonal functions, not so well known as the Fourier series, is that defined by:

$$f_n(t) = e^{-t/2} \sum_{r=0}^{n} \binom{n}{r} \cdot \left(-1\right)^r \cdot \frac{t^r}{r!} \tag{32}$$

in which $\binom{n}{r} = nC_r = \dfrac{n!}{r!\,(n-r)!}$

These are known as Laguerre functions and are in fact *orthonormal* over the range 0 to $\infty$ i.e.

$$\left.\begin{aligned} \int_0^\infty f_m(t).f_n(t).dt &= 0 \text{ if } m \neq n; \\ &= 1 \text{ if } m = n \end{aligned}\right\} \tag{33}$$

Thus the coefficients, $c_m$, indicated in a series expansion such as equation (21):

$$g(t) = \sum_{m=0}^{\infty} c_m . f_m(t) \tag{21 bis}$$

that used Laguerre functions would be given simply by (c.f. equation (23)):

$$c_m = \int_0^\infty g(t) \, . \, f_m \, (t) \, . \, dt \tag{34}$$

Equation (32) can be written:

$$f_n(t) = e^{-t/2} \cdot L_n(t) \tag{35}$$

where $$L_n(t) = \sum_{r=0}^{n} \binom{n}{r} \left(-1\right)^r \frac{t^r}{r!} \tag{36}$$

The $L_n(t)$ are called Laguerre *polynomials* and can be shown to obey the recurrence formula:

$$L_{n+1}(t) = \frac{1}{n+1} \left\{ (2n + 1-t).\ L_n(t) - n.L_{n-1}(t) \right\} \tag{37}$$

Thus the higher Laguerre polynomials (and so the higher Laguerre *functions*) can be built up successively. The first few polynomials are:

$$L_0(t) = 1$$
$$L_1(t) = 1 - t$$
$$L_2(t) = 1 - 2t + \frac{1}{2}t^2$$
$$L_3(t) = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3$$

etc.

DOOGE (1965) has studied the use of Laguerre functions in deriving the impulse response of time-invariant linear systems. Using equation (21), let an input, $x(t)$, to such a system have coefficients $a_m$, let the corresponding output, $y(t)$, have coefficients $A_m$ and let the impulse response, $h(t)$, have coefficients $\alpha_m$. DOOGE showed that the substitution of such series into the convolution integral (equation (7)) yields an equation for the $n^{th}$ order coefficient of the output:

$$A_n = \sum_{r=0}^{n} \alpha_r.a_{n-r} - \sum_{r=0}^{n-1} \alpha_r.a_{n-r-1} \tag{38}$$

If we now sum $A_0$, $A_1$ .... $A_n$ using equation (38), it will be seen that

$$\sum_{i=0}^{n} A_i = \sum_{r=0}^{n} \alpha_r.a_{n-r} \tag{39}$$

all other terms cancelling. We can thus take the term with $\alpha_n$ (when $r = n$) from the R.H.S. of (39) and arrive at:

$$\alpha_n = \frac{1}{a_o}\left[ \sum_{i=0}^{n} A_i - \sum_{r=0}^{n-1} \alpha_r . a_{n-r} \right] \tag{40}$$

In principle, then, given an $x(t)$ and the corresponding $y(t)$ for a time-invariant linear system, one can use equation (40) recursively to find one by one, starting from $\alpha_0$, all the coefficients of $h(t)$ having first found all the $a_i$ and $A_i$ coefficients from $x(t)$ and $y(t)$ via equation (34). As before, one can then either synthesise ordinates of $h(t)$ via equation (21) or, given a new (design storm) $x(t)$, derive the new $a_i$, find the $A_i$ coefficients of the new output from equation (38) and finally synthesise ordinates of that new output from equation (21).

In practice, as with the Fourier technique, there are restrictions on the Laguerre technique. The fact that catchment data can be sampled at only a finite number of data points means that we cannot formally integrate equation (34) to derive the Laguerre coefficients. Also, the Laguerre representation of each of the functions must be limited to a finite series.

However, DOOGE (1965) showed in his studies using synthetic data that numerical integration of (34) with an adequate number of data points gives acceptably close estimates for the $c_m$ coefficients and that a relatively small number of terms in the Laguerre function series suffices to give a good fit to the various functions he used.

In a recent private communication to the author, DOOGE has shown that there is a set of "discrete" Laguerre operations that yields the *TUH* as opposed to the continuous Laguerre operations described above (that lead to the *IUH*). This is a satisfying parallel to the harmonic series and Fourier series analysis described in 3.b.1 that also lead to the *TUH* and *IUH* respectively.

Figure 10 shows a flow diagram for a computer solution of the Laguerre approach to the inversion problem; in this, the *TUH* is shown as being got by numerical integration of the *IUH* not by the "discrete" Laguerre analysis mentioned above.

As pointed out by DOOGE (1965), the Laguerre functions method has the attraction of being physically meaningful. The factor $\dfrac{e^{-t/2}.t^r}{r!}$ of equation (32) can be seen to be of the same form as equation (12) (in fact, when divided by $2^{r+1}$, this factor will be seen to be the impulse response of a NASH-type cascade of $(r + 1)$ linear storages each having a storage characteristic

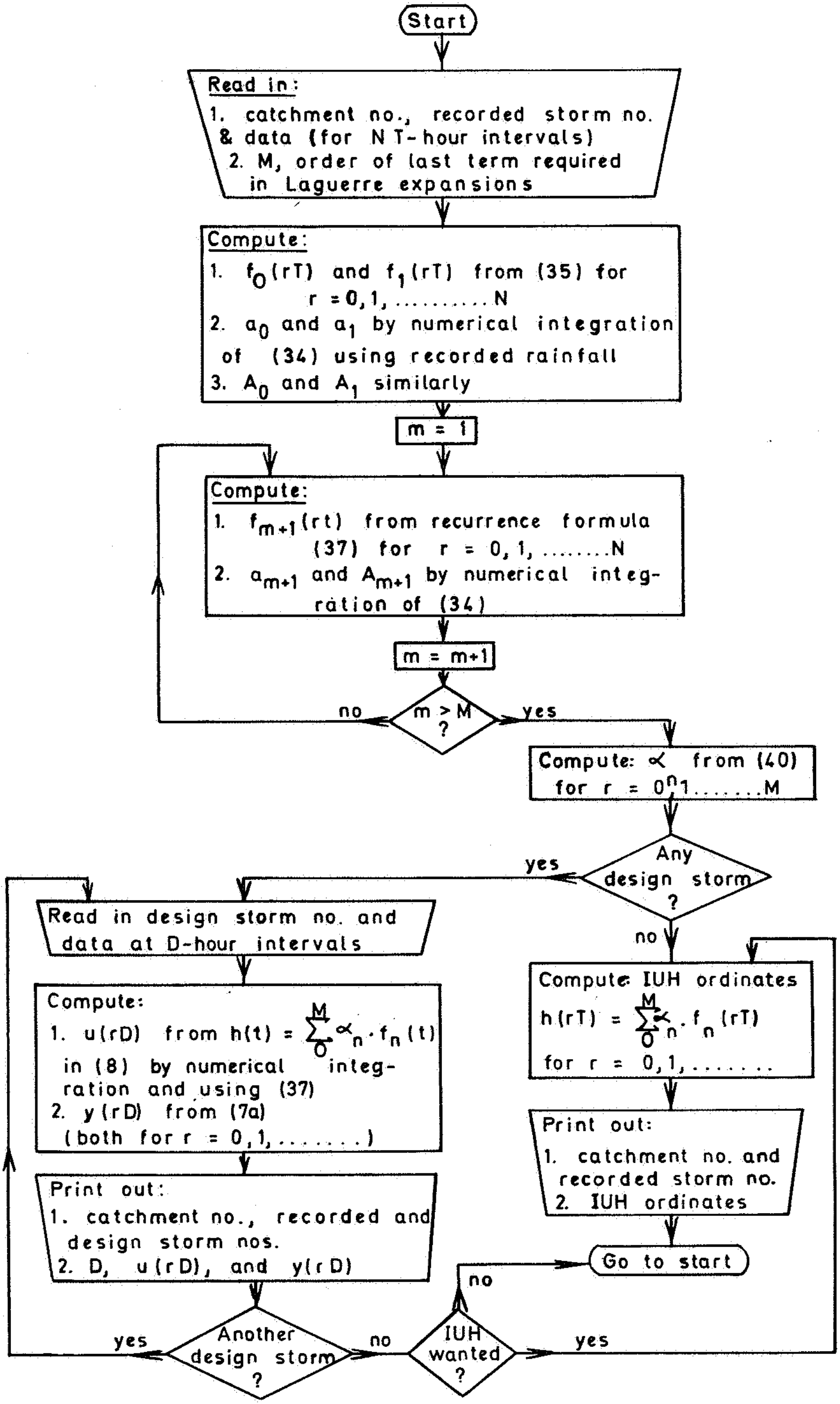


FIG. 10. Flow diagram for Laguerre technique.

$K = 2$ units). The terms of the Laguerre expansion of the impulse response can, in fact, provide a guide to a model of the catchment system. If that system actually consists of equal storages in parallel branches, each branch being a cascade, then the Laguerre technique would yield the size of the storages and the number in each branch. (It is necessary to find the size, $K$, by trial transformations of the real time scale with a factor $\frac{2}{K}$; this converts the real system to one with equal storages of size 2 units i.e. one suitable for Laguerre analysis.)

*b. 3. Inversion via matrices*

The conventional techniques of finite period unit hydrograph derivation were, for the most part, developed for desk-machine or graphical solution. One could convert these techniques to electronic computer use but such conversion may not yield the most efficient way of using a computer, which is much more than a high-speed calculating machine. It might be better to start with a fresh look at the basic process of unit hydrograph usage to see if we can find the best way of presenting the problem to an electronic computer.

This concept is exemplified in a T.V.A. study (1961) by SNYDER. Examination if the process involved in the "discrete" convolution of a rainfall excess with a *TUH* to produce a direct runoff hydrograph (equation (7a)) shows that basically the process is a multiplication of a matrix by a vector. Though this knowledge has little practical value in desk machine procedures, it is

$$
\begin{aligned}
y_1 &= X_1 \cdot u_1 + 0 + 0 + \dots\dots + 0 + 0 \\
y_2 &= X_2 \cdot u_1 + X_1 \cdot u_2 + 0 + \dots\dots + 0 + 0 \\
y_3 &= X_3 \cdot u_1 + X_2 \cdot u_2 + X_1 \cdot u_3 + 0 + \dots\dots + 0 + 0 \\
&\vdots \\
y_m &= X_m \cdot u_1 + X_{m-1} \cdot u_2 + \dots\dots + X_1 \cdot u_m + 0 + \dots\dots + 0 + 0 \\
y_{m+1} &= 0 + X_m \cdot u_2 + \dots\dots + X_2 \cdot u_m + X_1 \cdot u_{m+1} + 0 + \dots + 0 \\
&\vdots \\
y_{m+n-2} &= 0 + 0 + 0 + \dots\dots + 0 + X_m \cdot u_{n-1} + X_{m-1} \cdot u_n \\
y_{m+n-1} &= 0 + 0 + 0 + \dots\dots + 0 + 0 + X_m \cdot u_n
\end{aligned}
$$

FIG. 11. The equation relating rainfall and unit hydrograph ordinates to runoff.

highly relevant to the employment of digital electronic computing since such computing is very well suited to matrix algebra. A set of matrix subroutines is almost invariably a major part of the programme libraries associated with digital computers. Clearly then, one ought to recognise and make use of the basic unit hydrograph process (i.e. matrix multiplication) when employing a digital computer to derive a *TUH*.

As a bonus to this method of analysis, the matrix technique suggested in the T.V.A. study (1961) automatically provides a "least squares" solution for the *TUH*-ordinates. If there are $m$ volumes of rainfall excess $X_1, X_2 \ldots. X_m$ in successive $T$-hour periods and the *TUH* has $n$ ordinates $u_1, u_2 \ldots. u_n$, spaced $T$ apart, then the direct runoff hydrograph will have $(n + m - 1)$ ordinates $y_1, y_2 \ldots. y_{n+m-1}$, also spaced $T$ apart. In fact given the rainfall and runoff data and wanting the *TUH*-ordinates means solving $(n + m - 1)$ equations for $n$ unknowns (fig. 11). Clearly any technique of solution must incorporate a "best fit" device since any real data is bound to yield a set of incompatible equations. Any technique that does this automatically is obviously a good one to use.

The matrix equivalent of the equations of figure 11 is given in figure 12.

$$\begin{vmatrix}
X_1 & 0 & 0 & 0 & \cdots & & & & & 0 \\
X_2 & X_1 & 0 & 0 & \cdots & & & & & 0 \\
X_3 & X_2 & X_1 & 0 & \cdots & & & & & 0 \\
\vdots & & & & & & & & & \vdots \\
X_m & X_{m-1} & X_{m-2} & \cdots & X_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & X_m & X_{m-1} & \cdots & X_2 & X_1 & 0 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & \cdots & & & 0 & X_m & X_{m-1} \\
0 & 0 & 0 & 0 & \cdots & & & 0 & 0 & X_m
\end{vmatrix}
\times
\begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{vmatrix}
=
\begin{vmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_m \\ y_{m+1} \\ \vdots \\ y_{m+n-1} \end{vmatrix}$$

FIG. 12. The matrix form of the unit hydrograph procedure.

In the usual notation, the matrix equation can be written

$$|X| \, . \, |U| = |Y| \tag{42}$$

To solve this equation for $|U|$, one must first make the rectangular matrix $|X|$ a square one. This can be done by multiplying both sides of (42) by the transpose of $|X|$ viz. $|X|^T$, which is the matrix formed by interchanging the rows and columns of $|X|$.

Thus we get:

$$|X|^T . |X| . |U| = |Z| . |U| = |X|^T . |Y|$$

whence
$$|U| = |Z|^{-1} . |X|^T . |Y| \qquad (43)$$

Equation (43) gives the procedure for finding the *TUH*-ordinates directly from rainfall excess and direct runoff data using standard computer matrix routines. Hidden in the manipulations of the matrix algebra on the R.H.S. of (43) is the "least squares" curve-fitting technique mentioned above.

An extension of this technique to permit the evaluation of improved estimates of rainfall excess data (i.e. the determination of a "loss curve") was also represented in the T.V.A. study (1961). This extension uses the *TUH*-ordinates got from a first application of (43) to estimate errors in the first assessment of the rainfall data. Differences between the observed direct runoff data and the runoff computed with the first *TUH*-estimate and the initial rainfall data are used as the criterion of error. The improved rainfall data is now used to find a better estimate of the *TUH* and the whole process is repeated. Such an iterative procedure is, of course, admirably suited to a "loop" programme on a digital computer.

## 4. Non-linear treatments of catchment behaviour

Any account of computational methods that allow for non-linearity of catchment behaviour must draw very heavily from the publications of Amorocho and his collaborators (1961, 1963, 1964). Following Amorocho and Hart (1964) in classifying studies of the hydrological cycle, hydrological studies may be divided into the two areas of "physical hydrology" and "system investigations". The former deals with studies in the physical sciences of individual phenomena related directly or indirectly to the hydrological cycle. The latter is concerned with the investigation of hydrological systems considered as complete units in themselves, for the specific purpose of establishing quantitative relationships between precipitation and runoff. In physical hydrology the emphasis is on the *topics* of study while in system investigations it centres on the *methods* of study. The two areas are highly interdependent:

system investigations must rely on some knowledge of hydrological phenomena for a rational choice of system elements or techniques of analysis, while the demands of system investigations will focus attention on topics needing more detailed or intensive study.

The interest here lies in system investigations — more specifically in this section of the paper, in non-linear system investigations. Again, to recapitulate, a twofold classification can be made: general system analysis and general system synthesis. In the former, the relationship between input and output is established by a mathematical process that uses input and output data only, without any attempt at explicit description of the system mechanism. In the latter, the operation of the system is described by a combination of and linkages between components whose functions are known or assumed and whose presence in the system is presumed.

Although the non-linear system analysis approach holds great promise, it cannot yet be said to offer as practical and effective a working technique for hydrograph computation as does the synthesis approach even though the latter has certain weaknesses that will be difficult to eradicate. In consequence, the following accounts will deal rather more fully with the synthesis methods but more briefly with the analysis techniques.

### *a. Non-linear system synthesis*

Synthesis techniques usually begin with the postulation of a general conceptual model of catchment behaviour, the structure of the model and its functioning being based more or less subjectively on qualitative and semi-quantitative knowledge of the phenomena involved in the hydrological cycle. The basis of operation of such models is the principle of continuity, i.e. maintaining at all times a complete balance between all inputs, outputs and inner storage changes. On that basis, a general model is fitted to a specific catchment in some systematic way using recorded input and output data for that catchment. The fitting is done by making adjustments to the parameters governing the performance of the model until the output computed by the model when supplied with the recorded input agrees with the recorded output (within some specified tolerance).

Such quantitative models must inevitably be complex and yet they must be feasible to operate. Modern highspeed digital computers have provided the means of going a long way towards satisfying both these requirements; considerable progress in synthesis techniques in recent years has been made with the aid of computers. Before embarking on an account of two such techniques, some general cautions on the use of synthetic models are called for (a further discussion has been given by AMOROCHO and HART (1964)).

Clearly such models can only be as good as knowledge gained in the area of physical hydrology will permit; in many features they cannot as yet match even that imperfect knowledge. For example, it is usual to model the catchment mechanism as if it were a system with "lumped" input and output and "lumped" components whereas it is, of course, a "distributed" system with certainly a distributed input (variable areal rainfall pattern) and distributed flow processes even if the output at the catchment outlet may reasonably be treated as lumped. Another cause for care is the presence of errors in the recorded data used for adjusting a model: small data errors may result in large errors in those model parameters for which the model response is insensitive, thus leading to spurious conclusions. A third reason for caution arises from the considerable simplifications necessary in postulating a model structure in order that the model be workable: many of the fine details of the natural catchment system may get lost in the simplified model structure. Finally, although many different models might be adjusted to fit a given record equally well, there is, at present, no way of assessing which model is the 'best" one: until a technique for evaluating an optimum model structure is evolved, synthesis techniques must be viewed somewhat cautiously.

The two reviews of synthesis techniques presented below both postulate subjective catchment models of the type discussed above. The first summarises work on perhaps the most highly developed study yet made in which a quite sophisticated computer model has been fitted to more than thirty catchments of widely different types (and thereafter used as an aid to engineering design studies on those catchments). Its success has undoubtedly been largely due to the great experience, judgement and skill of its originators guiding their choice of parameter adjustments. The second review is of a more exploratory study using a much simpler model and mainly aimed at examining the possibility of employing an automatic objective technique of parameter adjustment.

### *a. 1. The Stanford watershed model*

LINSLEY and CRAWFORD at Stanford University have given full accounts (1962, 1964) of their notable computer study of streamflow synthesis. The Stanford watershed model aims at simulating the whole of the land phase of the hydrological cycle in any catchment. It is usually programmed to produce hourly streamflow data using hourly precipitation data and daily evapotranspiration data. For small catchments, shorter intervals may be used if the data is adequate. Full details being available in the references quoted, only a short summary of the Stanford model will be given.

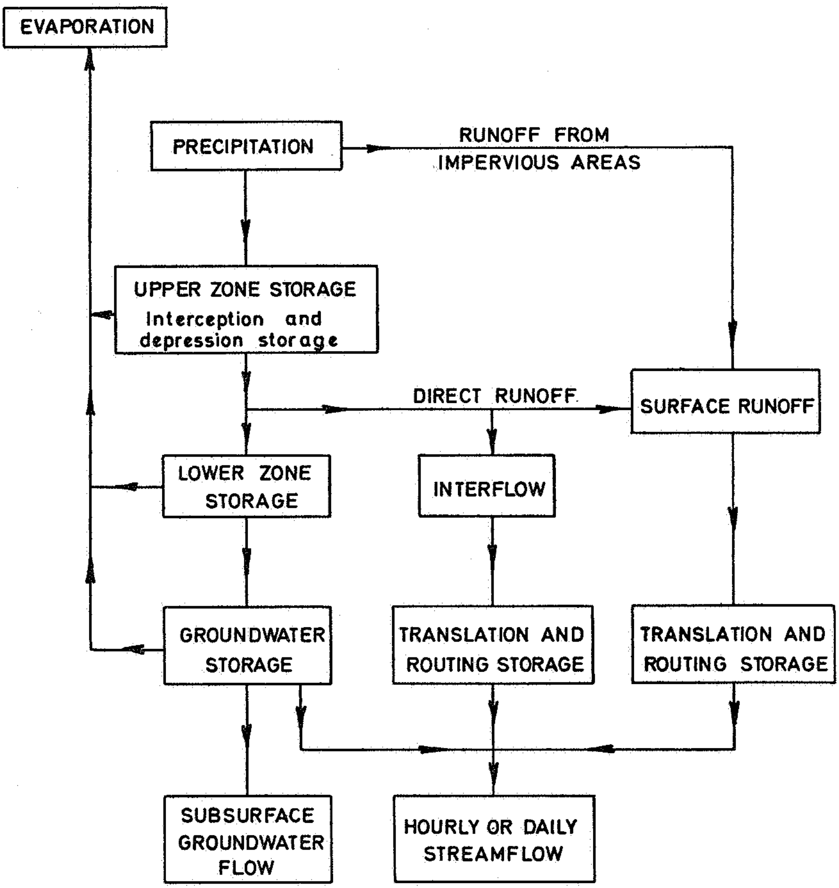


FIG. 13. The Stanford watershed model.

Figure 13 shows the general arrangement and components of the Stanford model. The characteristics of the component storage and routing elements are determined by relationships, expressed in terms of certain parameters, that represent as rationally as possible the behaviour of the various segments of the hydrological cycle.

After setting estimates of initial storage volumes hourly (or shorter) in-detailed knowledge of the elements of the hydrological cycle from studies in physical hydrology and the resulting more precise specification of their behavioural relationships will lead to more sophisticated, but inevitably more complicated, models than those used hitherto. Without denying the power and advantages of using engineering judgement and acquired skills, it is likely that adjustment of the larger number of parameters of more complex models by subjective trial and error procedures will become very difficult, if not impracticable.

Finding a set of "best fit" parameter values for given physical systems, given input and output data, is a frequently met problem in many fields of activity. Optimisation or "hill climbing" techniques have been developed that determine values of system parameters which maximise or minimise some function dependent on those parameters. Such techniques are completely objective. They make many useless tests of situations that would be dismissed out of hand by an experienced and skilled human investigator, but the tremendous speed with which a computer can make such tests compensates for such inefficiency.

In the catchment model context, an obvious parameter-dependent function to be optimised (minimised) is the difference between an observed outflow and the outflow computed by the model when supplied with the corresponding observed input. Other error criteria could be used (e.g. height or timing of peak flows) or, in fact, any combination of such criteria.

The author has been working on techniques for automatic optimisation of catchment model parameters and has published (DAWDY and O'DONNELL, 1965) some preliminary findings. Again only a brief account of this work will be given here.

The model postulated for these studies was deliberately made much simpler than the Stanford model but it follows the general outlines of figure 13. The emphasis was not on getting the "best" model but on the feasibility (or not) of the optimisation approach. Briefly, there were four storage elements whose behaviour and whose linkages were controlled by nine parameters in various simple control "rules". By assuming a long dry period prior to the start of a synthesis, all four initial storage values could be taken to be zero. To avoid the inevitable distortion due to the errors in real input and output data, the feasibility studies were carried out with sets of compatible error-free data. Such sets of data were synthesised with the model by allotting a set of values to the model parameters and finding the output generated by the model from an arbitrary input. Not only did this manoeuvre provide error-free data — the "correct" parameter values for each set of compatible input/output data were known a priori. Thus, with an input to the computer consisting of a crements of rainfall are entered into the model. The incoming rainfall either becomes direct runoff or is detained in upper and lower soil moisture storages, the latter feeding a groundwater storage. The three storage zones combine to represent the effects of highly variable soil moisture profiles and groundwater conditions. The upper zone storage absorbs a large part of the first few hours of rain in a storm. The lower zone storage controls long-term infiltration. The groundwater storage controls baseflow in the stream. Evaporation is permitted at the potential rate from the upper zone storage and at less than the potential rate from the lower zone storage and groundwater storage.

The direct runoff is split into two components, surface runoff and interflow, which have separate translation and routing procedures. Total streamflow is the sum of surface runoff, interflow and baseflow.

In applying the model, the typical procedure is to select a five to six year portion of rainfall and runoff records for a catchment. This period is used to develop estimates of the model parameters that fit the general model to the given catchment. A second period of record is then used as a control to check the accuracy of the parameters obtained from the first period. Comparison in the control period is based on such things as total monthly flows, daily flow duration curves, and hourly hydrographs of the two maximum floods each water year.

The numerical values both of the initial volumes in the storages and of the parameters that control the operation of the various model components are selected on the basis of previous experience and on a judgement of what is considered reasonable. Some of the parameter and initial storage values can be estimated quite closely simply from a preliminary study of the runoff records (e.g. recession curves for groundwater storage characteristics) or by choosing the end of a dry spell as a starting point (soil moisture storage at or near zero). Adjustment of the parameter values during the fitting stage is done in two ways. Most are adjusted by the operator, via a combination of experience and intuition, using clues provided by the timing and magnitude of the difference between the synthesised and recorded streamflow hydrographs. Some of the parameters are evaluated by the computer itself, using an internal looping routine of successive approximations.

*a. 2. Automatic parameter adjustment*

The successful operation of a digital computer model in which the parameter values are adjusted by the operator relies to a considerable extent on the skilled experience and personal judgement of that operator. Increases in compatible set of input and output data and an initial set of parameter values displaced from the "correct" set, both the absolute progress and the rate of progress of the parameter values towards that correct set could be observed and measured during any optimisation run.

The paper referred to gives some results of using a modified version of an optimisation procedure developed by ROSENBROOK (1960). The basis of this procedure consists of a search in an $n$-dimensional vector space formed by $n$ orthogonal axes ($n$ being the number of parameters being optimised) until some function, $U$, dependent on the parameters is optimised.

The search is made in a number of recursive stages. During each stage, movement is made along each orthogonal axis in a series of steps. A step of arbitrary length $e$, is attempted first. This is treated as a success if the resulting new $U$ value is an improvement on or equal the previous $U$ value. If a success, the step $e$ is allowed and a new step $ae$ is attempted (where $a > 1$); if a failure, the step $e$ is not allowed and a new step $\beta e$ is attempted (where $-1 < \beta < 0$) (in the end an attempt must succeed for each axis because $e$ becomes so small after repeated failures that it causes no change in $U$). The stage is terminated as soon as at least one successful step followed by one step has been made along each of the $n$ axes.

At the end of each stage, a new set of $n$ orthogonal axes for the next stage is evaluated. This reorientation of axes is based on the magnitudes of the movements made along the $n$ axes of the current stage in such a way that the first component of the new system of orthogonal axes lies along the direction of fastest advance in the current stage. Only for the first stage do the orthogonal axes lie along the parameter axes.

For the catchment model, $U$ was defined as the sum of the squares of the differences between the recorded (compatible) output and the output generated with the current set of parameter values; the optimum value of $U$ was thus zero. The same quantity was used to examine the *sensitivity* of the model response to each of nine parameters by finding the $U$ value computed with eight "correct" parameters but with the ninth displaced by 1% from its correct value.

It was found that, as applied to the catchment model, there was a tendency for the optimisation search to stagnate after a number of stages but that the search could be stimulated again by starting a new "round" of stages with the arbitrary steps, $e$, set back to their start-of-run values.

Table 1 shows some results of a typical optimisation run for a certain record and, in the final column, the 1% sensitivity of the model for the "correct" parameter values of that record.

The $U$ values (and sensitivity figures) as defined above have been transfor-

TABLE 1

| Parameter No. | Correct value | Starting value | OPTIMISATION End-of-round values Round 1 | Round 2 | Round 4 | Round 6 | Residual difference (%) | 1% sensitivity |
|---|---|---|---|---|---|---|---|---|
| 1 | 10.0 | 15.0 | 10.17 | 10.13 | 10.011 | 10.015 | 0.15 | 0.58 |
| 2 | 0.2 | 0.1 | 0.1721 | 0.1700 | 0.1973 | 0.1972 | 1.4 | 0.26 |
| 3 | 2.0 | 3.0 | 2.931 | 2.113 | 1.983 | 1.970 | 1.5 | 0.12 |
| 4 | 2.0 | 1.0 | 1.952 | 1.943 | 1.972 | 1.967 | 1.7 | 0.16 |
| 5 | 2.0 | 1.0 | 1.815 | 1.886 | 1.936 | 1.947 | 2.7 | 0.16 |
| 6 | 40.0 | 35.0 | 31.32 | 57.10 | 45.17 | 43.96 | 9.9 | 0.045 |
| 7 | 0.1 | 0.15 | 0.3059 | 0.2615 | 0.1174 | 0.1143 | 14.0 | 0.026 |
| 8 | 4.0 | 6.0 | 5.834 | 18.03 | 19.82 | 19.27 | 380 | 0.012 |
| 9 | 0.1 | 0.15 | 2.049 | 0.6363 | 0.5282 | 0.5574 | 460 | 0.012 |
| $\frac{\text{R.M.S. error}}{\text{Mean Q}} \times 100\%$ | | 18.4 | 0.78 | 0.28 | 0.060 | 0.044 | | |

med in table 1 into root-mean-square values expressed as a percentage of the mean runoff of the given record. This dimensionless mean form is possibly more directly informative than the absolute multi-step sum of squares, $U$, which is a criterion adequate only as a measure of optimisation progress.

The main conclusions to be drawn from table 1 are:

(1) the greater the sensitivity of the model response to a parameter, the closer and sooner will that parameter be optimised;

(2) the less sensitive parameters have insignificant influence on the fitting of a record and could be dropped from the model (at least for that record);

(3) the final matching of the output, to within 0.05% on average, is extremely close considering the mean initial error of nearly 20%.

These results and conclusions are based on synthetic error-free data and the use of one optimisation technique. A further study using synthetic data to which controlled amounts of "noise" have been added will be necessary before there can be any test with real data. Other studies comparing the speed of convergence of different techniques have been started.

Rapid progress is being made by applied mathematicians in providing improved optimisation techniques capable of handling larger numbers of parameters and having rapid convergence. Modern computers are several orders of magnitude larger in storage and faster in operation than those of even a few years ago. As studies in physical hydrology permit more faithful modelling of catchment behaviour, the inevitably more complex models that result should be capable of automatic fitting by optimisation techniques.

*b. Non-linear system analysis*

What such analysis aims at is the determination of the system response function from recorded input and output data. This is analogous to the inversion problem of the unit hydrograph method. However, general methods of direct non-linear inversion are not yet available. Some indirect approximation of the inversion operation has to be used.

One such method examined by AMOROCHO (1963) is the use of a functional series representation. The convolution integral (equation (7)) for a linear system written in the form:

$$y(t) = \int_0^t h_1(\tau) \cdot x\,(t-\tau) \cdot d\tau$$

can be called the "functional" of that system. This is the most elementary case of a general functional:

$$f_n(t) = \int_0^t \int_0^t \ldots \int_0^t h_n\,(\tau_1\,,\,\tau_2\,,\,\ldots\,\tau_n) \cdot x(t-\tau_1) \cdot x(t-\tau_2) \ldots x(t-\tau_n) \cdot d\tau_1 \cdot d\tau_2 \ldots d\tau_n \qquad (44)$$

which is a multi-dimensional generalisation of the convolution integral.

In the functional series approach, it is assumed that the action of a complex non-linear system is equivalent to the summation of the actions of separate elemental systems of progressively higher order and described by (44) with $n = 1, 2 \ldots.$

$$\text{i.e. } y(t) = \sum_{n=1}^{\infty} f_n\,(t) \qquad (45)$$

is taken to be an equivalent functional series representation of non-linear systems. The first term of the series (45) is the familiar linear convolution: the kernel $h_1(\tau)$ operates one-dimensionally on all elements of the input $x(t-\tau)$ with no interaction between these elements i.e. the kernel has a unique value for each $\tau$ value and the operation at time $\tau$ is unaffected by input elements at other times. For the other terms in the series the $h_n$ $(\tau_1, \tau_2 \ldots \tau_n)$ are called the "system functions" and are multidimensional kernels which, along with the products $x(t-\tau_1).x(t-\tau_2) \ldots x(t-\tau_n)$ indicate that interaction exists between elements of input occurring at different times.

Pending the arrival of techniques, still under development, that will permit the derivation of multiple system functions by direct inversion using complex input and output data, AMOROCHO (1961, 1963) investigated various simple input situations, both analytically and with laboratory catchment experiments. Much remains to be done before the considerable advantages of general non-linear analysis techniques will be available in standardised procedures of hydrograph computation but there can be little doubt that the pioneering work to date of AMOROCHO and his colleagues is going to yield substantial future rewards. The main advantages of the non-linear system analysis approach are:

(1) freedom from subjective bias insofar as nothing need be known or specified (in the physical sense) about the system;

(2) it is not necessary that the system should satisfy physical continuity conditions between total input, total output and inner storage, i.e. one can operate with recorded precipitation and recorded streamflow without having to account for evapotranspiration, loss of ground water from the catchment etc.

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