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Simple stochastic differential equations to analyze non-linear hydrological systems

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Time series models in Hydrology

- ARIMA models
- Transfer-function noise (TFN) models
- TFN models with physical basis (Salas and Smith, 1987: streamflow; Parlange et al., 1992: soil moisture; Knotters and Bierkens, 2000: groundwater)
- PIRFICT approach (Von Asmuth and co-authors): TFN models with continuous impulse response function and noise model.

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Time series models in Hydrology

Most hydrological models are about the water balance

$$\frac{dS}{dt} = P - E(S) - Q(S)$$

and some (non-)linear storage-loss equation

$$\left.\begin{aligned} E &= E_p S^\beta \\ Q &= \varepsilon S^\gamma \end{aligned}\right\} \frac{dS}{dt} = P - E_p S^\beta - \varepsilon S^\gamma$$

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Time series models in Hydrology

Add some noise (for instance runoff coefficient is random:)

$$\varepsilon = <\varepsilon> + \varepsilon' \rightarrow \frac{dS}{dt} = P - E_p S^\beta - <\varepsilon> S^\gamma + S^\gamma \varepsilon'$$

Yields a stochastic differential equation:

$$dS = \left(P - E_p S^\beta - <\varepsilon> S^\gamma\right) dt + S^\gamma dW_t$$

with $\varepsilon' dt \equiv \sigma dW_t$

$$<dW_t> = 0 \quad <dW_t dW_s> = \begin{cases} dt & t = s \\ 0 & t \neq s \end{cases}$$

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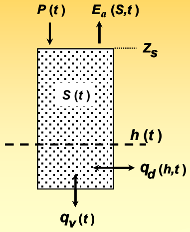
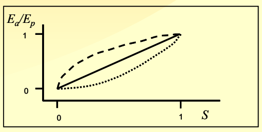
Example 1: shallow water table dynamics



$$S(t) = \int_{h(t)}^{z_s} \left[\frac{\theta(z,t) - \theta_r}{\theta_s - \theta_r} \right] dz$$

$$E_a(S,t) = F_c E_p(t) [S(t)]^c$$



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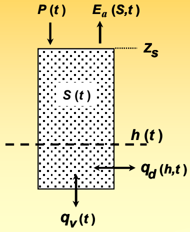
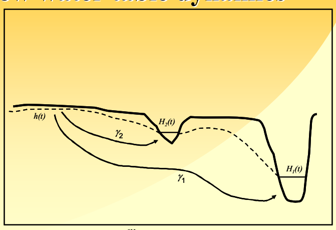
Example 1: shallow water table dynamics





$$q_d(h,t) = \sum_{i=1}^{m_d} q_{d_i}(h,t)$$

$$q_{d_i}(h,t) = \frac{h(t) - H_i(t)}{\gamma_i}$$

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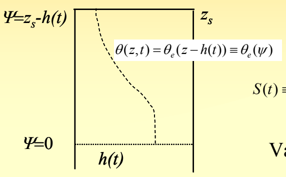
Example 1: shallow water table dynamics

Soil water balance:

$$\left[\varepsilon_0 + (\theta_s - \theta_r)\right]\frac{dh}{dt} + (\theta_s - \theta_s)\frac{d\left[(z_s - h(t))S(t)\right]}{dt} = p(t) - E_a(t) + q_v(t) - q_d(h,t)$$



Need: $S(t)$ -> $S(h)$:

$$S(t) \equiv S(h(t)) = \frac{1}{z_s - h(t)} \int_0^{z_s - h(t)} \left[\frac{\theta_e(\psi) - \theta_r}{\theta_s - \theta_r}\right] d\psi$$

Van Genuchten:

$$S(h(t)) = \frac{1}{\alpha(z_s - h(t))}\left[1 + \left[\alpha(z_s - h(t))\right]^{-n}\right]^{\frac{-1}{n}}$$

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Example 1: shallow water table dynamics

Yields differential equation:

$$G(h)\frac{dh}{dt} = P(t) - E_a(S(h),t) + q_v(t) - q_d(h,t)$$

With $G(h)$ (storage coefficient) given by:

$$G(h) = \varepsilon_0 + (\theta_s - \theta_r)\left|1 - \left[1 + \left[\alpha(z_s - h)\right]^{-n}\right]^{\frac{-1}{n}}\right|$$

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Example 1: shallow water table dynamics

Adding noise:

$$G(h)\frac{dh}{dt} = P(t) - E_a(S(h),t) + q_v(t) - q_d(h,t) + \sigma\xi(t)$$

Yields the stochastic differential equation:

$$dh = a(h,t)dt + b(h)dW_t$$

$$a(h,t) = \frac{P(t) - E_a(S(h),t) + q_v(t) - q_d(h,t)}{G(h)}$$

$$b(h) = \frac{\sigma}{G(h)}$$

$$< dW_t >= 0$$

$$< dW_t dW_s >= \begin{cases} dt & t = s \\ 0 & t \neq s \end{cases}$$

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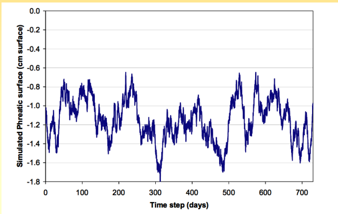
Example 1: shallow water table dynamics

Solution:

a)Simulating realizations with stochastic integration



b)Or calculating probability density $f_h(h,t|h_0,t_0)$ (Fokker-Planck Equation)

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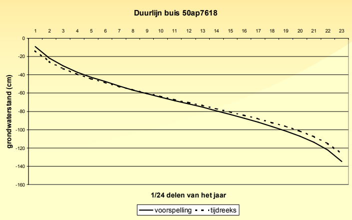
Example 1: shallow water table dynamics

If $P(t)$ is constant: steady state solution:

$$f_h(h) = \frac{N}{b^2(h)}\exp\left[2\int_{-\infty}^{h}\frac{a(h')}{b^2(h')}\right]dh'$$



Calculating depth-duration lines

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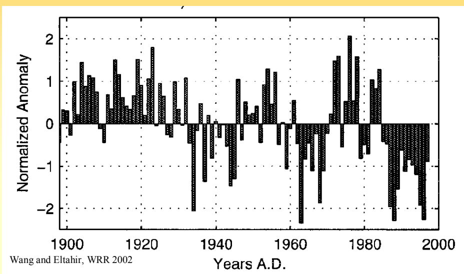
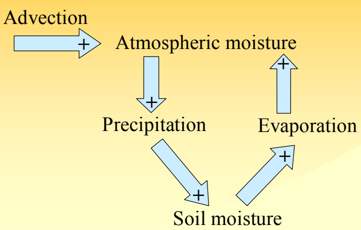
Example 2: continental scale land-atmosphere feedbacks

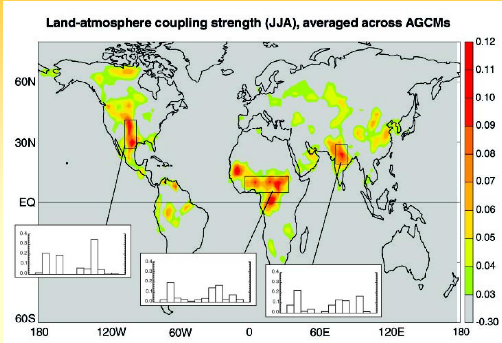
Average yearly rainfall anomalies over the Sahel



Presumed mechanisms



Feedback hotspots:

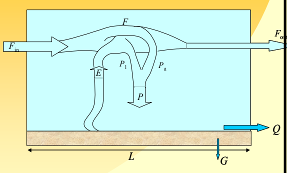


Koster et al., Science, 2004

A simple stochastic model

Entekhabi et al., 1991; Budyko, 1974.

$$ds_t = a(s_t)dt + b(s_t)dW_t$$



Feedback parameter: $\alpha = \dfrac{LE_p}{2wu}$

$$a(s_t) = \frac{P_a}{nz_r}(1 + <\alpha> s_t^c)(1 - \varepsilon s_t^r) - \frac{E_p}{nz_r} s_t^c$$

$$b(s_t) = \frac{P_a}{nz_r} s_t^c (1 - \varepsilon s_t^r)\sigma$$

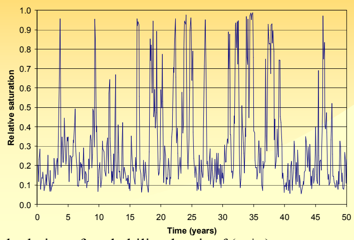
$$<dW_t> = 0$$

$$<dW_t dW_s> = \begin{cases} dt & t = s \\ 0 & t \neq s \end{cases}$$

A simple stochastic model

Solutions to Stochastic Differential Equation:

1. Simulation of realisation of stochastic process s_t: (stochastic integration)



2. Calculation of probability density $f_S(s,t|t_0)$ (Fokker-Planck Equation)

A simple stochastic model

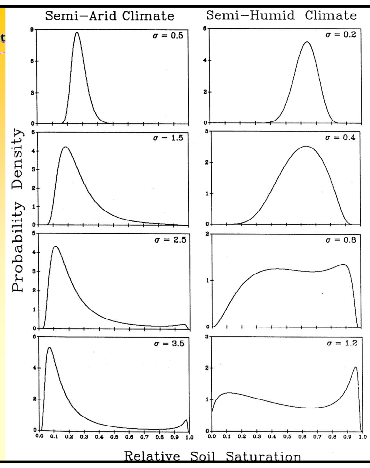
Probability density function:

$$f_S(s) = C\exp\left(-2\ln b(s) + \int_0^s \frac{2a(s)}{\sigma^2 b(s)}ds\right)$$

Parameters:

TABLE 1. Summary of model parameters for the semiarid and semihumid climatic cases.

Parameters	Units	Semiarid	Semihumid
c	[·]	1	0.5
ϵ	[·]	1	1
r	[·]	6	6
nZ_r	[m]	0.5	1.2
u	[ms^{-1}]	4	4
w	[m]	0.01	0.04
L	[m]	2.5×10^6	2.5×10^6
E_p	[m yr^{-1}]	2.2	1.5
P_a	[m yr^{-1}]	0.4	1.0



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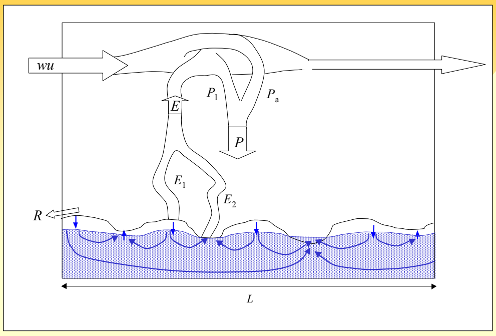
Another mechanism: groundwater convergence



Bierkens and Van den Hurk, Geophysical Research Letters 2007

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A simple stochastic model

$$ds_1 = A(s_1, s_2)dt + B(s_1, s_2)dW_t$$

$$ds_2 = C(s_1, s_2)dt + D(s_1, s_2)dW_t$$

$$A(s_1,s_2) = \frac{P_\mathrm{a}}{nz_\mathrm{r}}\{1 + <\alpha> [(1-f_\mathrm{g})s_1^c + f_\mathrm{g}s_2^c]\}(1-\varepsilon s_1^r) - \frac{E_\mathrm{p}}{nz_\mathrm{r}}s_1^c - k_\mathrm{s}s_1^b$$

$$B(s_1,s_2) = \frac{P_\mathrm{a}}{nz_\mathrm{r}}[(1-f_\mathrm{g})s_1^c + f_\mathrm{g}s_2^c](1-\varepsilon s_1^r)\sigma$$

$$C(s_1,s_2) = \frac{P_\mathrm{a}}{nz_\mathrm{r}}\{1 + <\alpha> [(1-f_\mathrm{g})s_1^c + f_\mathrm{g}s_2^c]\}(1-\varepsilon s_2^r) - \frac{E_\mathrm{p}}{nz_\mathrm{r}}s_2^c + q_\mathrm{g}(\bar{s}_1)$$

$$D(s_1,s_2) = \frac{P_\mathrm{a}}{nz_\mathrm{r}}[(1-f_\mathrm{g})s_1^c + f_\mathrm{g}s_2^c](1-\varepsilon s_2^r)\sigma$$

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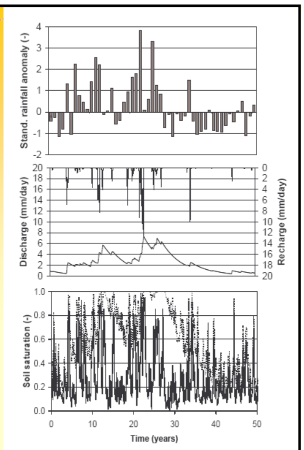
Groundwater convergence modelled as a linear reservoir:

$$q_\mathrm{g}(\bar{s}_1, t) = \left(\frac{1-f_g}{f_g}\right)k_s \frac{1}{J}\int_{-\infty}^{t} s_1^b(t-\tau)e^{-\tau/J}d\tau$$

$$J = \frac{S_y L^2}{\pi^2 kD}$$

Result: Stochastic delayed differential equation

$s_1(t)$, $s_2(t)$ are not necessarily stationary processes



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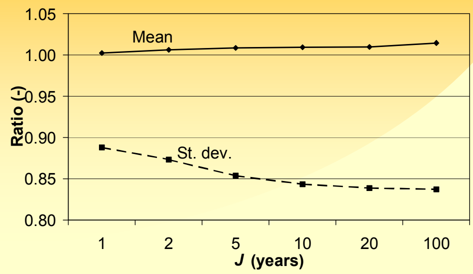
Yearly rainfall statistics: $f_g = 0.10$



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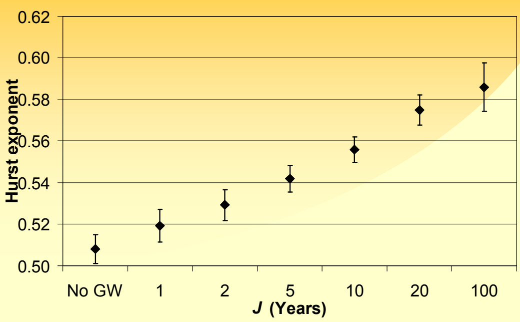
Persistance: Hurst exponent



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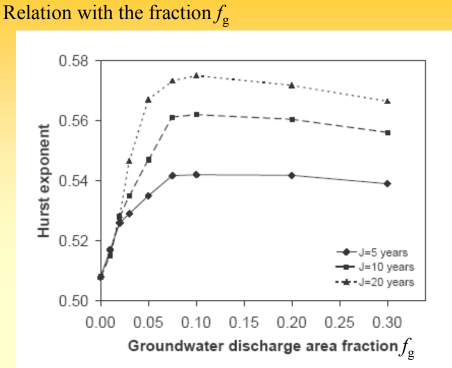
Relation with the fraction f_g



Conclusions

- Compared to no groundwater and with increasing residence time
  - mean *P* increases (but not much)
  - variance *P* decreases
  - persistance *P* increases
- As a function of groundwater discharge area:
  - mean *P* first increases after which it decreases

Speculation: Long term memory in river discharge partly caused by long term persistance in rainfall due to groundwater stores

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Benefits of SDEs

- Natural way of dealing with non-linear dynamics
- Diffence with discrete representation becomes important for extreme values of non-linear systems under noise
- Quasi-analytical solutions for (conditional) pdf available.